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Extension Theory for Braided-Enriched Fusion Categories

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For a braided fusion category $\mathcal{V}$, a $\mathcal{V}$-fusion category is a fusion category $\mathcal{C}$ equipped with a braided monoidal functor $\mathcal{F} : \mathcal{V} \to Z(\mathcal{C})$. Given a fixed $\mathcal{V}$-fusion category $(\mathcal{C}, \mathcal{F})$ and a fixed G-graded extension $\mathcal{C} \subseteq \mathcal{D}$ as an ordinary fusion category, we characterize the enrichments $\widetilde{\mathcal{F}} : \mathcal{V} \to Z(\mathcal{D})$ of $\mathcal{D}$ that are compatible with the enrichment of $\mathcal{C}$. We show that G-crossed extensions of a braided fusion category $\mathcal{C}$ are G-extensions of the canonical enrichment of $\mathcal{C}$ over itself. As an application, we parameterize the set of G-crossed braidings on a fixed G-graded fusion category in terms of certain subcategories of its center, extending Nikshych's classification of the braidings on a fusion category.

1 Introduction

In previous articles [36, 37], we defined monoidal categories enriched in a braided monoidal category $\mathcal{V}$ and showed this notion was equivalent to an oplax, strongly unital, braided monoidal functor from $\mathcal{V}$ into the Drinfeld center of an ordinary monoidal category. When the functor $\mathcal{F}^Z : \mathcal{V} \to Z(\mathcal{C})$ is strong monoidal, this coincides with the notion of a 1-morphism $\mathcal{V} \to \mathsf{Vec}$ in a suitable Morita 4-category [5] (see also Section 2.3 below), and with the module tensor categories of [26]. The recent work of Kong and Zheng uses monoidal categories enriched in a braided category to give a unified treatment of

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gapped and gapless edges for 2D topological orders [7, 30, 31]. Of particular importance is the case where $\mathcal{V}$ is a braided fusion category and $\mathcal{F}^z : \mathcal{V} \to Z(\mathcal{C})$ is a braided strong monoidal functor into the Drinfeld center of another fusion category $\mathcal{C}$. We call such a pair $(\mathcal{C}, \mathcal{F}^z)$ a *$\mathcal{V}$-fusion category*.

The extension theory for fusion categories of [19] has proven to be an immensely important tool. Particular applications include the process of gauging a global symmetry on a modular tensor category [2, 9], permutation symmetries on modular tensor categories [20], rank finiteness for (G-crossed) braided fusion categories [28], and classification theorems for tensor categories generated by an object of small dimension [16, 17].

In this article, we define the notion of a G-graded extension of a $\mathcal{V}$-fusion category. We begin by proving that G-gradings on a fusion category $\mathcal{C}$ are equivalent to liftings of a fixed fiber functor $\mathsf{Rep}(G) \to \mathsf{Vec} = \langle 1_{\mathcal{C}} \rangle \subseteq \mathcal{C}$ to $Z(\mathcal{C})$. Fixing such a G-grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, we see that an object $(c, \sigma_{\cdot,c}) \in Z(\mathcal{C})$ satisfies $c \in \mathcal{C}_e$ if and only if $(c, \sigma_{\cdot,c})$ lies in the Müger centralizer $\mathsf{Rep}(G)'$. Given this, we define a G-graded $\mathcal{V}$-fusion category to be a $\mathcal{V}$-fusion category $(\mathcal{C}, \mathcal{F}^z)$ such that the underlying fusion category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ is G-graded and $\mathcal{F}^z(\mathcal{V}) \subseteq \mathsf{Rep}(G)' \subset Z(\mathcal{C})$.

Theorem 1.1. Fix a G-graded extension $\mathcal{C} \subseteq \mathcal{D}$ of ordinary fusion categories and a $\mathcal{V}$-fusion category structure $(\mathcal{C}, \mathcal{F}^z)$ on $\mathcal{C}$. The following sets are in canonical bijection.

- For all $v \in \mathcal{V}$, extensions of the half-braiding for $\mathcal{F}^z(v)$ with $\mathcal{C}$ to a half-braiding with all of $\mathcal{D}$ coherently with respect to morphisms in $\mathcal{V}$.
- Lifts $\underline{\widetilde{\pi}} : \underline{G} \to \underline{\mathsf{BrPic}}^{\mathcal{V}}(\mathcal{C})$ of the monoidal 2-functor $\underline{\pi} : \underline{G} \to \underline{\mathsf{BrPic}}(\mathcal{C})$ afforded by the G-extension $\mathcal{D}$ (up to homotopy), where $\underline{\mathsf{BrPic}}^{\mathcal{V}}(\mathcal{C})$ is our newly introduced $\mathcal{V}$-enriched Brauer–Picard 2-groupoid (see Definition 4.3 below).
- Lifts $\widetilde{\mathcal{F}}^z : \mathcal{V} \to Z(\mathcal{C})^G$ such that $\mathrm{Forget}_G \circ \widetilde{\mathcal{F}}^z = \mathcal{F}^z$ where the categorical G-action $\underline{\rho} : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}^{br}(Z(\mathcal{C}))$ comes from the G-extension $\mathcal{C} \subseteq \mathcal{D}$ and $\mathrm{Forget}_G : Z(\mathcal{C})^G \to Z(\mathcal{C})$ forgets the G-equivariant structure.

This theorem characterizes the possible enrichments $\widetilde{\mathcal{F}}^z : \mathcal{V} \to Z(\mathcal{D})$ of $(\mathcal{C}, \mathcal{F}^z)$, which are compatible with the fixed G-graded extension $C \subseteq D$. The proof uses extension theory for fusion categories of [19] together with the results of [21].

Observe that the structures listed in Theorem 1.1 are more naturally viewed as collections of objects in higher groupoids rather than sets, and it would be more natural to prove an equivalence of groupoids rather than construct a bijection between these sets. However, each of these higher groupoids is in fact 0-*truncated*, that is, equivalent to a 0-groupoid, which is a set. We make this rigorous by showing homotopy fibers of certain forgetful functors are 0-truncated and equivalent to strict fibers. We discuss these notions in detail in Section 3 on homotopy fibers of forgetful functors.

Thus, one of our canonical bijections is typically constructed as a composite of bijections

$$\begin{aligned}\{\text{set } 1\} &\cong \{\text{strict fiber } 1\} \cong \tau_0\,\{\text{homotopy fiber } 1\} \cong \tau_0\,\{\text{homotopy fiber } 2\}\\ &\cong \{\text{strict fiber } 2\} \cong \{\text{set } 2\}\,,\end{aligned}$$

where τ_0 denotes taking the 0-truncation. This strategy is also employed to construct the canonical bijections asserted in Corollary 3.25 and Theorems 1.3, 4.5, 4.6, 5.9, and 7.2. We would like to emphasize that these results prove equivalences of cores of various higher categories, which happen to be 0-truncated, by providing a bijection on the 0-truncations. It would be interesting to see if some of these canonical bijections could be lifted to functorial constructions on the cores, or even on the higher categories.

The third description of compatible enrichments in Theorem 1.1 bears many similarities to the classification from [4] of G-equivariant structures on a connected étale algebra in a nondegenerately braided fusion category. Adapting the arguments and techniques from [4], we see that there are two *obstructions* to lifting our $\mathcal{V}$-enrichment. First, for every $g \in G$, we must have that $\mathcal{F}^Z \cong g \circ \mathcal{F}^Z$ as monoidal functors $\mathcal{V} \to Z(\mathcal{C})$. We call the existence of such monoidal natural isomorphisms for $g \in G$ the *first obstruction* to the equivariant functor lifting problem. When such monoidal natural isomorphisms exist, we say $\mathcal{D}$ *passes* the first obstruction or that the first obstruction *vanishes*. In this case, similar to [4], we show that lifts $\underline{\widetilde{\rho}} : \underline{G} \to \underline{\mathrm{Aut}}(Z(\mathcal{C})|\mathcal{F}^Z)$ correspond to splittings of a certain exact sequence.

Theorem 1.2. There is a short exact sequence

$$1 \longrightarrow \mathrm{Aut}_\otimes(\mathcal{F}^Z) \longrightarrow \mathrm{Aut}_\otimes(I \circ \mathcal{F}^Z) \longrightarrow G \longrightarrow 1, \tag{1.1}$$

where $I : Z(\mathcal{C}) \to Z(\mathcal{C})^G$ is the induction functor adjoint to the forgetful functor Forget_G (observe that while I is only oplax monoidal as an adjoint of a monoidal functor, it still makes sense to talk about the (oplax) monoidal automorphisms $\mathrm{Aut}_\otimes(I \circ \mathcal{F}^Z)$). Moreover, splittings of this exact sequence are in canonical bijection with lifts $\underline{\widetilde{\rho}} : \underline{G} \to \underline{\mathrm{Aut}}(Z(\mathcal{C})|\mathcal{F}^Z)$ as in the final case of Theorem 1.1.

We call the exact sequence (1.1) the *second obstruction* to the equivariant functor lifting problem. We say the second obstruction *vanishes* when this short exact sequence splits, and a splitting is a *witness* of the vanishing of the second obstruction. In Section 6, we calculate the splittings of (1.1) for various examples.

In Section 7, we give an application of our two main theorems above to extend Nikshych's [40] classification of braidings on a fixed fusion category, classifying G-crossed braidings on a fixed G-graded fusion category in Theorem 7.3. The main tool is the following theorem, which extends [3, Proposition 2.4] in the unitary setting.

Theorem 1.3. Let $\mathcal{V}$ be a braided fusion category and $\mathcal{C}$ a G-graded extension of $\mathcal{V}$ as fusion categories. The set of extensions of the self-enrichment $\mathcal{V} \to Z(\mathcal{V})$ to $Z(\mathcal{C})$ characterized in Theorem 1.1 is in bijective correspondence with equivalence classes of G-crossed braidings on $\mathcal{C}$.

We then describe the equivalence classes of G-crossed braidings on group theoretical G-graded fusion categories, for example, $\mathsf{Vec}(H, \omega)$ and $\mathsf{Rep}(H)$ for appropriate groups H, in terms of group theoretical data.

2 Background

In this article, we assume the reader is familiar with tensor categories, in particular the book [18]. We typically use their conventions. For example, the Drinfeld center of a tensor category $\mathcal{C}$ has objects $(c, \sigma_{\bullet,c})$ where $c \in \mathcal{C}$ and $\sigma_{\bullet,c} = \{\sigma_{a,c} : a \otimes c \to c \otimes a\}_{a\in\mathcal{C}}$ is a family of half-braidings. In this convention, the braiding on $Z(\mathcal{C})$ is given by $\beta_{(c,\sigma_{\bullet,c}),(d,\tau_{\bullet,d})} := \tau_{c,d} : c \otimes d \to d \otimes c$. When $\mathcal{C}$ is a monoidal subcategory of a monoidal category $\mathcal{D}$, we use the notation $Z_{\mathcal{C}}(\mathcal{D})$ for the relative Drinfeld center. This agrees with the notation of [21] but is the reverse of the notation of [24].

2.1 Braided enriched monoidal categories

Recall from [29] that given a monoidal category $\mathcal{V}$, a $\mathcal{V}$-category $\mathcal{C}$ has objects together with hom objects $\mathcal{C}(a \to b) \in \mathcal{V}$ for all $a, b \in \mathcal{C}$. For every $a, b, c \in \mathcal{C}$, we have a composition morphism $- \circ_{\mathcal{C}} - \in \mathcal{V}(\mathcal{C}(a \to b)\mathcal{C}(b \to c) \to \mathcal{C}(a \to c))$ that satisfies an associativity axiom. For every $a \in \mathcal{C}$, we have an identity element $j_a \in \mathcal{V}(1_{\mathcal{V}} \to \mathcal{C}(a \to a))$ that satisfies a unitality axiom.

There are also notions of $\mathcal{V}$-functors and ($1_\mathcal{V}$-graded) $\mathcal{V}$-natural transformations. We refer the reader to [29] for more details. (See also the pedestrian exposition in [36, Section 2] or [37, Section 2].)

Definition 2.1. Given a $\mathcal{V}$-category $\mathcal{C}$, the *underlying category* $\mathcal{C}^\mathcal{V}$ has the same objects as $\mathcal{C}$, and the hom-sets are given by $\mathcal{C}^\mathcal{V}(a \to b) := \mathcal{V}(1_\mathcal{V} \to \mathcal{C}(a \to b))$. We leave the reader to work out the definitions of composition and identity morphisms for $\mathcal{C}^\mathcal{V}$.

Definition 2.2 ([34, 37]). A $\mathcal{V}$-category $\mathcal{C}$ is called *weakly tensored* if every representable functor $\mathcal{C}(a \to -) : \mathcal{C}^\mathcal{V} \to \mathcal{V}$ admits a left adjoint.

When $\mathcal{V}$ is closed, we can form the self-enrichment $\widehat{\mathcal{V}}$ of $\mathcal{V}$ over itself [29, Section 1.6]. In this case, the representable functor $\mathcal{C}(a \to -) : \mathcal{C}^\mathcal{V} \to \mathcal{V}$ can be promoted to a $\mathcal{V}$-functor $\widehat{\mathcal{C}}(a \to -) : \mathcal{C} \to \mathcal{V}$. A $\mathcal{V}$-category $\mathcal{C}$ is called *tensored* if every $\mathcal{V}$-representable functor $\widehat{\mathcal{C}}(a \to -) : \mathcal{C} \to \widehat{\mathcal{V}}$ admits a left $\mathcal{V}$-adjoint.

We now assume $\mathcal{V}$ is a braided monoidal category.

Definition 2.3. A (strict) $\mathcal{V}$-*monoidal category* is a $\mathcal{V}$-category $\mathcal{C}$ equipped with an associative monoid structure on objects, denoted ab for $a, b \in \mathcal{C}$, whose unit object is denoted by $1_\mathcal{C}$, together with a tensor product morphism $- \otimes_\mathcal{C} - \in \mathcal{V}(\mathcal{C}(a \to c)\mathcal{C}(b \to d) \to \mathcal{C}(ab \to cd))$ for all $a, b, c, d \in \mathcal{C}$ satisfying strict associativity and unitality axioms. The tensor product and composition morphisms must further satisfy the *braided interchange* relation

$$\begin{tikzcd}
\mathcal{C}(a \to b)\mathcal{C}(d \to e)\mathcal{C}(b \to c)\mathcal{C}(e \to f) \arrow[r, "(-\otimes_\mathcal{C}-)(-\otimes_\mathcal{C}-)"] \arrow[dd, "\mathrm{id}\beta_{\mathcal{C}(d\to e),\mathcal{C}(b\to c)}\mathrm{id}"] & \mathcal{C}(ad \to be)\mathcal{C}(be \to cf) \arrow[d, "-\circ_\mathcal{C}-"] \\
& \mathcal{C}(ad \to cf) \\
\mathcal{C}(a \to b)\mathcal{C}(b \to c)\mathcal{C}(d \to e)\mathcal{C}(e \to f) \arrow[r, "(-\circ_\mathcal{C}-)(-\circ_\mathcal{C}-)"] & \mathcal{C}(a \to c)\mathcal{C}(d \to f). \arrow[u, "-\otimes_\mathcal{C}-"]
\end{tikzcd}$$

There are also notions of $\mathcal{V}$-monoidal functors and ($1_\mathcal{V}$-graded) $\mathcal{V}$-monoidal natural transformations. We refer the reader to [36, Section 2] or [37, Section 6.1] for more details.

Definition 2.4. A $\mathcal{V}$-monoidal category $\mathcal{C}$ is called *rigid* if its underlying monoidal category is rigid.

Remark 2.5. When a $\mathcal{V}$-monoidal category $\mathcal{C}$ is rigid, $\mathcal{C}$ is weakly tensored if and only if $\mathcal{C}(1_{\mathcal{C}} \to -) : \mathcal{C}^{\mathcal{V}} \to \mathcal{V}$ admits a left adjoint [37, Lemma 6.8]. When, in addition, $\mathcal{V}$ is rigid (which implies $\mathcal{V}$ is closed), $\mathcal{C}$ is tensored if and only if the $\mathcal{V}$-functor $\widehat{\mathcal{C}}(1_{\mathcal{C}} \to -)$ admits a left $\mathcal{V}$-adjoint [37, Corollary 7.3].

In [36], we proved a classification theorem for (weakly) tensored rigid $\mathcal{V}$-monoidal categories in terms of $\mathcal{V}$-module tensor categories [25]. The tensored case was treated in [37]. We now restrict our focus to the tensored case for ease of exposition, as all our examples in this article are tensored.

Definition 2.6. A *$\mathcal{V}$-module tensor category* consists of a pair $(\mathcal{T}, \mathcal{F}^Z)$ with $\mathcal{T}$ a monoidal category and $\mathcal{F}^Z : \mathcal{V} \to Z(\mathcal{T})$ a braided strongly monoidal functor. We call a $\mathcal{V}$-module tensor category

- *rigid* if $\mathcal{T}$ is rigid and
- *tensored* if the strong monoidal functor $\mathcal{F} := \mathrm{Forget}_Z \circ \mathcal{F}^Z$ admits a right adjoint.

Based on [26, Definition 3.2], a 1-morphism $(\mathcal{T}_1, \mathcal{F}_1^Z) \to (\mathcal{T}_2, \mathcal{F}_2^Z)$ of $\mathcal{V}$-module tensor categories consists of a pair (G, γ) where $G : \mathcal{T}_1 \to \mathcal{T}_2$ is a strong monoidal functor and $\gamma : \mathcal{F}_2 \Rightarrow G \circ \mathcal{F}_1$ is an *action coherence* monoidal natural isomorphism that satisfies the following compatibility with half-braidings:

$$\begin{tikzcd}
G(t) \otimes \mathcal{F}_2(v) \arrow[r, "\mathrm{id}\otimes\gamma_v"] \arrow[d, "\sigma_{G(t),\mathcal{F}_2(v)}"] & G(t) \otimes G(\mathcal{F}_1(v)) \arrow[r, "\cong"] & G(t \otimes \mathcal{F}_1(v)) \arrow[d, "G(\sigma_{t,\mathcal{F}_2(v)})"] \\
\mathcal{F}_2(v) \otimes G(t) \arrow[r, "\gamma_v\otimes\mathrm{id}"] & G(\mathcal{F}_1(v)) \otimes G(t) \arrow[r, "\cong"] & G(\mathcal{F}_1(v) \otimes t)
\end{tikzcd} \tag{2.1}$$

Based on [26, Definition 3.3], a 2-morphism $\kappa : (G, \gamma) \Rightarrow (H, \eta)$ between 1-morphisms $(\mathcal{T}_1, \mathcal{F}_1^Z) \to (\mathcal{T}_2, \mathcal{F}_2^Z)$ is a monoidal natural transformation $\kappa : G \Rightarrow H$ such that for all $v \in \mathcal{V}$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}_2(v) \arrow[rr, "\gamma_v"] \arrow[dr, "\eta_v"'] & & G(\mathcal{F}_1(v)) \arrow[dl, "\kappa_{\mathcal{F}_1(v)}"] \\
& H(\mathcal{F}_1(v)). &
\end{tikzcd} \tag{2.2}$$

We call an invertible 1-morphism between $\mathcal{V}$-module tensor categories an equivalence.

We have the following classification theorem, which has recently been extended to a 2-equivalence of 2-categories (pseudofunctor equivalence of bicategories) in [14].

Theorem 2.7 ([36, 37]). Let $\mathcal{V}$ be a braided monoidal category. There is a bijective correspondence between equivalence classes

$$\left\{\begin{array}{l}\text{Tensored rigid } \mathcal{V}\text{-monoidal}\\ \text{categories } \mathcal{C}\end{array}\right\} \cong \left\{\begin{array}{l}\text{Tensored rigid } \mathcal{V}\text{-module tensor}\\ \text{categories } (\mathcal{T}, \mathcal{F}^Z)\end{array}\right\}.$$

In light of Theorem 2.7, together with the results of [30, 31] in the fusion setting, we make the following definition.

Definition 2.8. A $\mathcal{V}$-*fusion category*, for $\mathcal{V}$ a braided fusion category, consists of a fusion category $\mathcal{C}$ together with a braided strong monoidal functor $\mathcal{F}^Z : \mathcal{V} \to Z(\mathcal{C})$. Observe as $\mathcal{F}^Z$ is a functor between fusion categories, it automatically admits a left adjoint, and hence $\mathcal{V}$-fusion categories are tensored.

We focus on the fusion setting in order to have access to the results of [19] and [21].

2.2 Extension theory for fusion categories

We rapidly review the results of [19] and [21] on extension theory for fusion categories.

Etingof–Nikshych–Ostrik [19] give a recipe for constructing G-extensions of a fixed fusion category $\mathcal{C}$ using cohomological obstruction theory.

Definition 2.9. Recall that a *categorical n-group* is an $(n+1)$-category with one 0-morphism such that every k-morphism is invertible up to a (k+1)-isomorphism for $k \leq n$, and all $(n+1)$-morphisms are invertible. Typically, we indicate the categorical group number by adding that number of underlines below. We denote the $k < n$ *truncation* obtained by inductively identifying higher isomorphism classes by simply removing underlines.

Example 2.10. Given a group G, we view it as a category with one object where every morphism is invertible. We get a categorical 1-group $\underline{G}$ by adding only identity 2-morphisms, and we get a categorical 2-group $\underline{\underline{G}}$ by only adding identity 2-morphisms to $\underline{G}$.

Example 2.11. Given a fixed fusion category $\mathcal{C}$, the 2-groupoid $\mathsf{Ext}(G,\mathcal{C})$ of G-extensions of $\mathcal{C}$ is the categorical 2-group whose unique object is $\mathcal{C}$, whose 1-morphisms are G-graded fusion categories $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ together with a fixed monoidal equivalence $I_{\mathcal{D}} : \mathcal{C} \to \mathcal{D}_e$, whose 2-morphisms are G-graded monoidal equivalences $F : \mathcal{D} \to \mathcal{E}$ together with a monoidal natural isomorphism $\alpha : I_{\mathcal{E}} \Rightarrow F \circ I_{\mathcal{D}}$, and whose 2-morphisms $(F_1, \alpha_1) \Rightarrow (F_2, \alpha_2)$ are monoidal natural equivalences $\gamma : F_1 \Rightarrow F_2$ such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "I_{\mathcal{D}}"] \arrow[dr, "I_{\mathcal{E}}"'] & \mathcal{D} \arrow[d, "F_2"] \arrow[dl, phantom, "\Nearrow\alpha_2"] \\ & \mathcal{E}. \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, "I_{\mathcal{D}}"] \arrow[dr, "I_{\mathcal{E}}"'] & \mathcal{D} \arrow[d, "F_1"'] \arrow[d, bend left=60, "F_2"] \arrow[dl, phantom, "\Nearrow\alpha_1"] \\ & \mathcal{E} \end{tikzcd} .$$

$$\gamma : F_1 \Rightarrow F_2$$

Remark 2.12. The 2-groupoid $\mathsf{Ext}(G,\mathcal{C})$ defined in the above example is equivalent to the one defined in [13, Definition 8.2] where each $\mathcal{I}_{\mathcal{D}} : \mathcal{C} \to \mathcal{D}_e$ is $\mathrm{id}_{\mathcal{C}}$ and each $\alpha : \mathrm{id}_{\mathcal{C}} \Rightarrow F|_{\mathcal{C}} \circ \mathrm{id}_{\mathcal{C}}$ is the identity monoidal natural isomorphism.

Example 2.13. Given a fixed fusion category $\mathcal{C}$, its *Brauer–Picard* groupoid $\underline{\mathsf{BrPic}}(\mathcal{C})$ is the categorical 2-group whose unique 0-morphism is $\mathcal{C}$, whose 1-morphisms are invertible $\mathcal{C}-\mathcal{C}$ bimodule categories, whose 2-morphisms are $\mathcal{C}-\mathcal{C}$ bimodule equivalences, and whose 3-morphisms are bimodule functor natural isomorphisms.

Definition 2.14. In Example 2.13 above, composition of $\mathcal{C}-\mathcal{C}$ bimodule categories is the relative Deligne tensor product. In more detail, suppose $\mathcal{D}$ is a fusion category, $\mathcal{M}_{\mathcal{D}}$ is a right $\mathcal{D}$-module category, and ${}_{\mathcal{D}}\mathcal{N}$ is a left $\mathcal{D}$-module category. The relative tensor product is a finitely semisimple category $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ together with a $\mathcal{D}$-balanced functor $\boxtimes_{\mathcal{D}} : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ satisfying the universal property that for every abelian category $\mathcal{P}$ and any $\mathcal{D}$-balanced functor $F : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{P}$, there exists a linear functor $\widetilde{F} : \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{P}$, unique up to unique natural isomorphism, such that the following diagram weakly commutes:

$$\begin{tikzcd} \mathcal{M} \boxtimes \mathcal{N} \arrow[d, "\boxtimes_{\mathcal{D}}"] \arrow[dr, "F"] & \\ \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\widetilde{F}"] & \mathcal{P}. \end{tikzcd}$$

When $\mathcal{M}$ is a left $\mathcal{C}$-module category and $\mathcal{N}$ is a right $\mathcal{E}$-module category, then $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ inherits the structure of a $\mathcal{C}-\mathcal{E}$ bimodule category. We refer the reader to [19, Section 3] for more details.

The following theorem classifies G-extensions via monoidal 2-functors $\underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$. For the definition of a monoidal 2-functor, see [13, Definition 2.8].

Theorem 2.15 ([19] and [13, Theorem 8.5]). Let $\mathcal{C}$ be a fusion category. There is an equivalence of 2-groupoids $\mathsf{Ext}(G, \mathcal{C}) \cong \mathrm{Hom}(\underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C}))$.

The main tool of [19] gives a cohomological prescription for constructing G-graded extensions by lifting a group homomorphism, or *symmetry action*, $\rho : G \to \mathsf{BrPic}(\mathcal{C})$ to $\underline{G} \to \underline{\mathsf{BrPic}}(\mathcal{C})$. We can lift ρ to a categorical action $\underline{\rho} : \underline{G} \to \underline{\mathsf{BrPic}}(\mathcal{C})$ if and only if the obstruction $o_3(\rho) \in H^3(G, \mathrm{Inv}(Z(\mathcal{C})))$ vanishes. In this case, the set of equivalence classes of liftings form a torsor over $H^2(G, \mathrm{Inv}(Z(\mathcal{C})))$. Given $\underline{\rho} : \underline{G} \to \underline{\mathsf{BrPic}}(\mathcal{C})$, there is a lift $\underline{\underline{\rho}} : \underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ if and only if the obstruction $o_4(\underline{\rho}) \in H^4(G, \mathbb{C}^\times)$ vanishes. In this case, the equivalence classes of liftings form a torsor over $H^3(G, \mathbb{C}^\times)$.

We now recall the main results of [21]. Suppose we have a G-extension $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ of $\mathcal{C}$. (Note that the convention $\mathcal{C} \subseteq \mathcal{D}$ is opposite to the convention of [21], which uses $\mathcal{D} \subseteq \mathcal{C}$.) The relative center $Z_{\mathcal{C}}(\mathcal{D})$ is canonically a *G-crossed braided extension* [18, Section 8.24] of $Z(\mathcal{C})$ whose G-equivariantization [18, Section 4.15] is equivalent to $Z(\mathcal{D})$. Moreover, the canonical equivalence $Z(\mathcal{D}) \cong Z_{\mathcal{C}}(\mathcal{D})^G$ intertwines both forgetful functors to $Z_{\mathcal{C}}(\mathcal{D})$ and maps $\mathsf{Rep}(G)' \subset Z(\mathcal{D})$ to $Z(\mathcal{C})^G$ up to a canonical monoidal natural isomorphism.

$$\begin{tikzcd}
Z(\mathcal{C})^G \arrow[rr, leftrightarrow, "\cong"] \arrow[d, hookrightarrow] & & \mathsf{Rep}(G)' \arrow[d, hookrightarrow] \\
Z_{\mathcal{C}}(\mathcal{D})^G \arrow[rr, leftrightarrow, "\cong"] \arrow[dr, "\mathrm{Forget}_{\mathcal{C}}"'] & \arrow[d, phantom, "\curvearrowright"] & Z(\mathcal{D}) \arrow[dl, "\mathrm{Forget}_G"] \\
& Z_{\mathcal{C}}(\mathcal{D}) &
\end{tikzcd} \tag{2.3}$$

2.3 The 4-category of braided tensor categories

By [23, 27], there is a 4-category of braided tensor categories BrTens, and the sub-4-category BrdFus of braided fusion categories is 4-dualizable by [5, Theorem 1.19].

Following [5], we now describe the n-morphisms and the composition operations of the 4-category BrdFus.

- 0-morphisms are braided fusion categories.
- 1-morphisms $\mathsf{BrdFus}_1(\mathcal{A} \to \mathcal{B})$ are multifusion categories $\mathcal{C}$ together with a braided monoidal functor $F_{\mathcal{C}} : \mathcal{A} \boxtimes \mathcal{B}^{\mathrm{rev}} \to Z(\mathcal{C})$ called a *central structure*.

Sometimes, we denote $\mathcal{C} \in \mathsf{BrdFus}_1(\mathcal{A} \to \mathcal{B})$ by ${}_{\mathcal{A}}\mathcal{C}_{\mathcal{B}}$.

The composite of ${}_{\mathcal{A}_1}\mathcal{C}_{\mathcal{A}_2}$ and ${}_{\mathcal{A}_2}\mathcal{D}_{\mathcal{A}_3}$ is defined as follows. First, we look at the Deligne tensor product $\mathcal{C}\boxtimes\mathcal{D}$, which comes equipped with a braided monoidal functor $F : \mathcal{A}_2^{\mathrm{rev}} \boxtimes \mathcal{A}_2 \to Z(\mathcal{C} \boxtimes \mathcal{D})$. We define $\mathcal{C} \boxtimes_{\mathcal{A}_2} \mathcal{D}$ to be $(\mathcal{C} \boxtimes \mathcal{D})_L$, the category of left L-modules in $\mathcal{C} \boxtimes \mathcal{D}$, where $L \in \mathcal{A}_2^{\mathrm{rev}} \boxtimes \mathcal{A}_2$ is the commutative algebra obtained by taking $I(1_{\mathcal{A}_2})$, where I is the left adjoint to the canonical tensor product functor $\otimes : \mathcal{A}_2^{\mathrm{rev}} \boxtimes \mathcal{A}_2 \to \mathcal{A}_2$, given by $\otimes(a \boxtimes b) := a \otimes b$ and using the braiding for the tensorator. This algebra is commutative since $\otimes$ is a central functor [11, Lemma 3.5]. If $\mathcal{A}_2$ is nondegenerate, this algebra is identified with the canonical Lagrangian algebra under the standard equivalence $\mathcal{A}_2^{\mathrm{rev}} \boxtimes \mathcal{A}_2 \cong Z(\mathcal{A}_2)$. To see that $\mathcal{C} \boxtimes_{\mathcal{A}_2} \mathcal{D}$ has the structure of a 1-morphism in $\mathsf{BrdFus}_1(\mathcal{A}_1 \to \mathcal{A}_3)$, we observe that $Z((\mathcal{C} \boxtimes \mathcal{D})_L) \cong Z(\mathcal{C} \boxtimes \mathcal{D})_L^{loc}$, the L-local modules in $Z(\mathcal{C} \boxtimes \mathcal{D}) \cong Z(\mathcal{C}) \boxtimes Z(\mathcal{D})$ by [11, Theorem 3.20]. Since $\mathcal{A}_1$ centralizes $F_{\mathcal{A}_2^{\mathrm{rev}}}(\mathcal{A}_2^{\mathrm{rev}})\boxtimes Z(\mathcal{D})$ and $\mathcal{A}_3^{\mathrm{rev}}$ centralizes $Z(\mathcal{C})\boxtimes F_{\mathcal{A}_2}(\mathcal{A}_2)$ in $Z(\mathcal{C})\boxtimes Z(\mathcal{D})$, we get a braided monoidal functor $\mathcal{A}_1 \boxtimes \mathcal{A}_3^{\mathrm{rev}} \to Z(\mathcal{C} \boxtimes \mathcal{D})_L^{loc} \cong Z((\mathcal{C} \boxtimes \mathcal{D})_L)$.

An explicit example calculation of the composite $\mathrm{Ad}E_8 \boxtimes_{\mathsf{Fib}} \mathrm{Ad}E_8'$ appears in [41].

- The 2-morphisms $\mathsf{BrdFus}_2(\mathcal{C}, \mathcal{D})$ are finitely semisimple $\mathcal{C} - \mathcal{D}$ bimodule categories $\mathcal{M}$ together with natural isomorphisms $\eta_{a,m} : m \triangleleft F_{\mathcal{D}}(a) \to F_{\mathcal{C}}(a) \triangleright m$ for $a \in \mathcal{A}\boxtimes\mathcal{B}^{\mathrm{rev}}$ and $m \in \mathcal{M}$ called a $\mathcal{A}\boxtimes\mathcal{B}^{\mathrm{rev}}$-*centered structure* such that the following diagrams commute (here, we suppress names of arrows):

$$\begin{tikzcd}
 & F_{\mathcal{C}}(a) \triangleright (c \triangleright m) \arrow[r] & (c \triangleright m) \triangleleft F_{\mathcal{D}}(a) \arrow[dr] & \\
(F_{\mathcal{C}}(a) \otimes c) \triangleright m \arrow[ur] \arrow[dr] & & & c \triangleright (m \triangleleft F_{\mathcal{D}}(a)) \\
 & (c \otimes F_{\mathcal{C}}(a)) \triangleright m \arrow[r] & c \triangleright (F_{\mathcal{C}}(a) \triangleright m) \arrow[ur] &
\end{tikzcd} \tag{2.4}$$

$$\begin{tikzcd}
 & F_{\mathcal{C}}(a) \triangleright (m \triangleleft d) \arrow[r] & (m \triangleleft d) \triangleleft F_{\mathcal{D}}(a) \arrow[dr] & \\
(F_{\mathcal{C}}(a) \triangleright m) \triangleleft d \arrow[ur] \arrow[dr] & & & m \triangleleft (d \otimes F_{\mathcal{D}}(a)) \\
 & (m \triangleleft F_{\mathcal{D}}(a)) \triangleleft d \arrow[r] & m \triangleleft (F_{\mathcal{D}}(a) \otimes d) \arrow[ur] &
\end{tikzcd} \tag{2.5}$$

$$\begin{tikzcd}
F_{\mathcal{C}}(a \otimes b) \triangleright m \arrow[r] \arrow[d] & m \triangleleft F_{\mathcal{D}}(a \otimes b) \arrow[r] & m \triangleleft (F_{\mathcal{D}}(a) \otimes F_{\mathcal{D}}(b)) \arrow[d] \\
(F_{\mathcal{C}}(a) \otimes F_{\mathcal{C}}(b)) \triangleright m \arrow[d] & & (m \triangleleft F_{\mathcal{D}}(a)) \triangleleft F_{\mathcal{D}}(b) \arrow[d] \\
F_{\mathcal{C}}(a) \triangleright (F_{\mathcal{C}}(b) \triangleright m) \arrow[r] & F_{\mathcal{C}}(a) \triangleright (m \triangleleft F_{\mathcal{D}}(b)) \arrow[r] & (F_{\mathcal{C}}(a) \triangleright m) \triangleleft F_{\mathcal{D}}(b)
\end{tikzcd} \tag{2.6}$$

The definitions of horizontal and vertical compositions of 2-morphisms are given in [5, pp. 41 and 42]. For our purposes, we need to know that vertical composition is the relative Deligne tensor product ${}_{\mathcal{C}}\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}_{\mathcal{E}}$ discussed earlier in Definition 2.14. As described in [5, Definition and Proposition 3.13], when $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are equipped with central structures $F_{\mathcal{C}}, F_{\mathcal{D}}, F_{\mathcal{E}}$, respectively, and $\mathcal{M}, \mathcal{N}$ are equipped with $\mathcal{A} \boxtimes \mathcal{B}^{\text{rev}}$-centered structures $\eta^{\mathcal{N}}, \eta^{\mathcal{M}}$ satisfying (2.4), (2.5), (2.6), the $\mathcal{C} - \mathcal{E}$ bimodule category $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ is equipped with the $\mathcal{A} \boxtimes \mathcal{B}^{\text{rev}}$-centered structure

$$m \boxtimes_{\mathcal{D}} (n \triangleleft F_{\mathcal{E}}(a)) \cong m \boxtimes_{\mathcal{D}} (F_{\mathcal{D}}(a) \triangleright n) \cong (m \triangleleft F_{\mathcal{D}}(a)) \boxtimes_{\mathcal{D}} n \cong (F_{\mathcal{C}}(a) \triangleright m) \boxtimes_{\mathcal{D}} n. \tag{2.7}$$

- Let $\mathcal{M}$ and $\mathcal{N}$ be two 2-morphisms with source $\mathcal{C}$ and target $\mathcal{D}$. Then, a 3-morphism is a bimodule functor $G : \mathcal{M} \to \mathcal{N}$ such that the following diagram commutes:

$$\begin{tikzcd}
G(m \triangleleft F_{\mathcal{D}}(a)) \arrow[r, "G(\eta_{a,m})"] \arrow[d] & G(F_{\mathcal{C}}(a) \triangleright m) \arrow[d] \\
G(m) \triangleleft F_{\mathcal{D}}(a) \arrow[r, "\eta_{a,G(m)}"] & F_{\mathcal{C}}(a) \triangleright G(m).
\end{tikzcd} \tag{2.8}$$

- The 4-morphisms are bimodule natural transformations with no extra compatibility required!

Remark 2.16. Observe that we may consider a fusion category $\mathcal{C} \in \mathsf{BrdFus}_1(\mathsf{Vec} \to \mathsf{Vec})$ where we suppress the obvious braided central functor $\mathcal{F}^z : \mathsf{Vec} \to Z(\mathcal{C})$. Then, $\underline{\mathsf{BrPic}}(\mathcal{C})$ is exactly the core (consisting of only the invertible morphisms) of the endomorphism 3-category $\mathrm{End}^{123}(\mathcal{C})$ that has

- a single 0-morphism $\mathcal{C}$,
- 1-morphisms $\mathsf{BrdFus}_2(\mathcal{C} \to \mathcal{C})$,
- 2-morphisms the 3-morphisms in BrdFus, and
- 3-morphisms the 4-morphisms in BrdFus.

Remark 2.17. Observe that given a $\mathcal{V} \in \mathsf{BrdFus}$, a 1-morphism $(\mathcal{C}, \mathcal{F}^Z) \in \mathsf{BrdFus}_1(\mathcal{V} \to \mathsf{Vec})$ is exactly a $\mathcal{V}$-fusion category.

Recall that nondegenerate braided fusion categories $\mathcal{A}, \mathcal{B}$ are said to be *Witt equivalent* [11, Definition 5.1 and Remark 5.2] if there exist multifusion categories $\mathcal{C}, \mathcal{D}$ such that $\mathcal{A} \boxtimes Z(\mathcal{C}) \cong \mathcal{B} \boxtimes Z(\mathcal{D})$. We conclude this section with the following observation.

Theorem 2.18. Suppose $\mathcal{A}, \mathcal{B}$ are nondegenerate braided fusion categories and $\mathcal{C} \in \mathsf{BrdFus}_1(\mathcal{A} \to \mathcal{B})$. The following statements are equivalent.

1. $\mathcal{C}$ is an invertible 1-morphism in BrdFus.
2. $F_{\mathcal{C}} : \mathcal{A} \boxtimes \mathcal{B}^{\text{rev}} \to Z(\mathcal{C})$ is a braided equivalence.

Before proving the theorem, we observe that the existence of $\mathcal{C}$ as in (2) above is equivalent to the Witt equivalence of $\mathcal{A}$ and $\mathcal{B}$ by [11, Remark 5.2 and Corollary 5.8].

Proof. Suppose $\mathcal{C}$ is an invertible 1-morphism in $\mathsf{BrdFus}(\mathcal{A} \to \mathcal{B})$. First, since $\mathcal{A}$ and $\mathcal{B}^{\text{rev}}$ are nondegenerate, every braided tensor functor out of $\mathcal{A} \boxtimes \mathcal{B}^{\text{rev}}$ is fully faithful. Hence, $Z(\mathcal{C}) \cong \mathcal{A} \boxtimes \mathcal{D}_1 \boxtimes \mathcal{B}^{\text{rev}}$ for some nondegenerate braided fusion category $\mathcal{D}_1$. Let $\mathcal{C}^{-1} \in \mathsf{BrdFus}(\mathcal{B} \to \mathcal{A})$ be an inverse for $\mathcal{C}$ such that $\mathcal{A} \cong (\mathcal{C} \boxtimes \mathcal{C}^{-1})_L$ as 1-morphisms in $\mathsf{BrdFus}_1(\mathcal{A} \to \mathcal{A})$, where $L \in \mathcal{B}^{\text{rev}} \boxtimes \mathcal{B}$ is the canonical Lagrangian algebra. By a similar argument as before, $Z(\mathcal{C}^{-1}) \cong \mathcal{B} \boxtimes \mathcal{D}_2 \boxtimes \mathcal{A}^{\text{rev}}$ for some nondegenerate braided fusion category $\mathcal{D}_2$. Observe now that

$$Z(\mathcal{C} \boxtimes \mathcal{C}^{-1}) \cong Z(\mathcal{C}) \boxtimes Z(\mathcal{C}^{-1}) \cong \mathcal{A} \boxtimes \mathcal{D}_1 \boxtimes \mathcal{B}^{\text{rev}} \boxtimes \mathcal{B} \boxtimes \mathcal{D}_2 \boxtimes \mathcal{A}^{\text{rev}}.$$

This means that by

$$Z(\mathcal{C} \boxtimes \mathcal{C}^{-1})_L^{loc} \cong \mathcal{A} \boxtimes \mathcal{D}_1 \boxtimes \mathcal{D}_2 \boxtimes \mathcal{A}^{\text{rev}}.$$

But since $Z(\mathcal{C} \boxtimes \mathcal{C}^{-1})_L^{loc} \cong Z((\mathcal{C} \boxtimes \mathcal{C}^{-1})_L) \cong Z(\mathcal{A}) \cong \mathcal{A} \boxtimes \mathcal{A}^{\text{rev}}$ as $\mathcal{A}$ is nondegenerate, we must have $\mathcal{D}_1$ and $\mathcal{D}_2$ are trivial, and thus $Z(\mathcal{C}) \cong \mathcal{A} \boxtimes \mathcal{B}^{\text{rev}}$.

Conversely, if $Z(\mathcal{C}) \cong \mathcal{A} \boxtimes \mathcal{B}^{\text{rev}}$, then observe that $Z(\mathcal{C}^{\text{mp}}) \cong \mathcal{B} \boxtimes \mathcal{A}^{\text{rev}}$, where $\mathcal{C}^{\text{mp}}$ is the monoidal opposite of $\mathcal{C}$. For the canonical Lagrangian algebra $L \in \mathcal{B}^{\text{rev}} \boxtimes \mathcal{B}$,

$$(\mathcal{C} \boxtimes \mathcal{C}^{\text{mp}})_L \cong (\mathcal{A} \boxtimes \mathcal{B}^{\text{rev}} \boxtimes \mathcal{B} \boxtimes \mathcal{A}^{\text{rev}})_L \cong \mathcal{A} \boxtimes \mathcal{A}^{\text{rev}} \cong Z(\mathcal{A}),$$

and so $(\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{mp}})_L \cong \mathcal{A}$ as 1-morphisms in $\mathsf{BrdFus}_1(\mathcal{A} \to \mathcal{A})$. Similarly, we have that $(\mathcal{C}^{\mathrm{mp}} \boxtimes \mathcal{C})_{L'} \cong \mathcal{B}$ as 1-morphisms in $\mathsf{BrdFus}_1(\mathcal{B} \to \mathcal{B})$, where L' is the canonical Lagrangian algebra in $\mathcal{A}^{rev} \boxtimes \mathcal{A}$. ■

3 Truncation of Homotopy Fibers and Classification of G-Gradings on a Fusion Category

In this article, we will often discuss various notions that are somewhat evil from a categorical perspective, such as classifying lifts of a fixed functor or G-gradings on a fixed fusion category. In this section, we discuss how to make these notions not evil by using the notion of truncation of a homotopy fiber. In the cases we care about the most, we can show that the homotopy fiber of a particular (2-)functor truncates to a set, and this set is in canonical bijection with a strict "set theoretic" fiber.

As an example, in Section 3.3 below, we classify the *set* of G-gradings on a fixed fusion category $\mathcal{T}$ in terms of fully faithful $\mathsf{Rep}(G)$-fibered enrichments.

3.1 Homotopy fibers of forgetful functors

In this section, we make sense of how various structures on a fixed monoidal category, like G-gradings for a fixed group G, or braidings, form a *set* and not a category.

Grothendieck's *Homotopy Hypothesis* posits that homotopy n-types are equivalent to n-groupoids for all $n \in \mathbb{N} \cup \{\infty\}$ via the fundamental groupoid construction. In this section, we use the term *n-groupoid* as a synonym for *homotopy n-type* and *weak n-functor* for *a homotopy class of continuous maps*.

Fact 3.1. For weak n-categories, the homotopy hypothesis is known to hold for $n \leq 3$ [33] to various degrees. We will only use it for $n \leq 2$ for this article. In more detail,

- the strict 2-category of groupoids, functors, and natural transformations is equivalent to the 2-category of homotopy 1-types, continuous maps, and homotopy classes of homotopies;
- by [35], the homotopy category of strict 2-groupoids and strict 2-functors localized at the strict 2-equivalences is equivalent to the 1-category of homotopy 2-types and homotopy classes of continuous maps;
- by [33], the homotopy category of Gray-groupoids and Gray-functors localized at the Gray-equivalences is equivalent to the 1-category of homotopy 3-types and homotopy classes of continuous maps.

Definition 3.2. Recall that the path space and homotopy fiber construction produces a fibration from any continuous map of spaces. We now explain this in the language of n-groupoids.

Suppose $\mathcal{C}, \mathcal{D}$ are n-groupoids and $U : \mathcal{C} \to \mathcal{D}$ is an n-functor. The *path space* of U, denoted $\mathsf{Path}(U)$, has objects triples (c, d, ψ) with $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $\psi \in \mathcal{C}(U(c) \to d)$ an isomorphism, $(c_1, d_1, \psi_1) \to (c_2, d_2, \psi_2)$ are triples (A, B, α) with $A \in \mathcal{C}(c_1 \to c_2)$, $B \in \mathcal{D}(d_1 \to d_2)$, and $\alpha \in \mathcal{D}(\psi_2 \circ U(A) \Rightarrow B \circ \psi_1)$ is a 2-isomorphism, and so forth, where k-morphisms consist of triples of a k-morphism in $\mathcal{C}$, a k-morphism in $\mathcal{D}$, and a $(k+1)$-isomorphism in $\mathcal{D}$ compatible with lower structure. Here, we interpret an $(n+1)$-isomorphism as an equality.

The *homotopy fiber* of U at $d \in \mathcal{D}$, denoted $\mathsf{hoFib}_d(U)$, has objects pairs (c, ψ) with $c \in \mathcal{C}$ and $\psi \in \mathcal{D}(U(c) \to d)$ an isomorphism, 1-morphisms $(c_1, \psi_1) \to (c_2, \psi_2)$ are pairs (A, α) with $A \in \mathcal{C}(c_1 \to c_2)$ and $\alpha : \mathcal{D}(\psi_1 \Rightarrow \psi_2 \circ U(A))$ is a 2-isomorphism, and so forth, where k-morphisms consist of pairs of a k-morphism in $\mathcal{C}$ and a $(k+1)$-isomorphism in $\mathcal{D}$ compatible with lower structure. Here, we interpret an $(n+1)$-isomorphism as an equality.

Definition 3.3. Suppose $\mathcal{C}, \mathcal{D}$ are n-groupoids and $U : \mathcal{C} \to \mathcal{D}$ is a weak n-functor. We call U *k-truncated* or $(k+1)$-*monic* [1, Section 5.5] if

- $k = n$: no condition;
- $k = n - 1$: faithful on n-morphisms;
- $k = n - 2$: fully faithful on n-morphisms;
- $-2 \leq k < n - 2$: fully faithful on n-morphisms and essentially surjective on j-morphisms for all $k+2 \leq j \leq n-1$. (Thus, a (-2)-truncated n-functor is an equivalence.)

Under the homotopy hypothesis, U being n-truncated corresponds to $U_* : \pi_*(\mathcal{C}) \to \pi_*(\mathcal{D})$ being injective on $\pi_{k+1}(\mathcal{C})$ and an isomorphism on $\pi_j(\mathcal{C})$ for all $j \geq k+2$ for all basepoints.

Proposition 3.4. Suppose $\mathcal{C}, \mathcal{D}$ are n-groupoids and $U : \mathcal{C} \to \mathcal{D}$ is a weak n-functor. For every $-2 \leq k \leq n$, U is k-truncated if and only if at each object $d \in \mathcal{D}$, the homotopy fiber $\mathsf{hoFib}_d(U)$ is k-truncated as an n-groupoid, that is, a k-groupoid (here, we use "negative categorical thinking" [1] when $k = -2, -1, 0$, i.e., a 0-groupoid is a set, a (-1)-groupoid is either a point or the empty set, and a (-2)-groupoid is a point).

Proof. Under the homotopy hypothesis, given a $d \in \mathcal{D}$, we have a fibration $\mathsf{hoFib}_d(U) \to \mathsf{Path}(U) \to \mathcal{D}$ that yields a long exact sequence in homotopy groups. Recall $\mathsf{hoFib}_d(U)$ is a k-groupoid if and only if $\pi_j(\mathsf{hoFib}_d(U)) = 0$ for all $j > k$. Since $\mathsf{Path}(U)$ is homotopy equivalent to $\mathcal{C}$, this happens if and only if $U_* : \pi_*(\mathcal{C}) \to \pi_*(\mathcal{D})$ gives an injection $\pi_{k+1}(\mathcal{C}) \hookrightarrow \pi_{k+1}(\mathcal{D})$ and an isomorphism on $\pi_j(\mathcal{C}) \cong \pi_j(\mathcal{D})$ for all $j \geq k+2$. The result now follows by quantifying over all objects $d \in \mathcal{D}$. ■

Remark 3.5. In this article, we will only every use the above proposition on n-functors between n-groupoids where $n \leq 2$, where the homotopy hypothesis is known to hold. Furthermore, it is straightforward to give an explicit proof of Proposition 3.4 for 2-groupoids using the formalism of bicategories that does not invoke the homotopy hypothesis. We leave these details to the interested reader.

3.2 Strictification of fibers

We now discuss for how to strictify the homotopy fiber of a 0-truncated functor $U : \mathcal{C} \to \mathcal{D}$ of n-groupoids for $n = 1, 2$.

3.2.1 $n = 1$

Suppose $\mathcal{C}, \mathcal{D}$ are groupoids and $U : \mathcal{C} \to \mathcal{D}$ is a 0-truncated functor, that is, a faithful functor. In the previous section, we saw that $U : \mathcal{C} \to \mathcal{D}$ gives a fibration $\mathsf{Path}(U) \to \mathcal{D}$. In order to strictify the homotopy fiber, it is necessary that $U : \mathcal{C} \to \mathcal{D}$ is a fibration in the canonical model structure on Cat, that is, an *isofibration*, meaning every isomorphism in $\mathcal{D}$ can be lifted to $\mathcal{C}$. Since U was assumed to be faithful, every isomorphism can be lifted *uniquely* subject to a fixed source and target.

However, an isofibration is not quite strong enough to strictify the homotopy fiber, as we may not have uniqueness of the source of the lifted isomorphism in $\mathcal{C}$. Hence, we require that

- U is a *discrete fibration*, that is, for every $c \in \mathcal{C}$ and every morphism $g \in \mathcal{D}(d \to U(c))$, there is a unique $b \in \mathcal{C}$ and a unique morphism $f \in \mathcal{C}(b \to c)$ such that $U(b) = d$ and $U(f) = g$.

Definition 3.6. We define the *strict fiber* $\mathsf{stFib}_d(U)$ of U at d is the set of objects $c \in \mathcal{C}$ such that $U(c) = d$.

Proposition 3.7. Suppose $U : \mathcal{C} \to \mathcal{D}$ is a 0-truncated functor of 1-groupoids, which is a discrete fibration. For every $d \in \mathcal{D}$, there is a canonical bijection $\mathsf{stFib}_d(U) \cong \tau_0(\mathsf{hoFib}_d(U)))$, the 0-truncation of the homotopy fiber of U at d.

Proof. Suppose $d \in \mathcal{D}$. If d is not in the essential image of U, then both $\mathsf{stFib}_d(U)$ and $\mathsf{hoFib}_d(U)$ are empty.

Now, assume d is in the essential image of U so that there is a $c \in \mathcal{C}$ and a $g \in \mathcal{D}(d \to U(c))$. Since U is a discrete fibration, there is a unique $b \in \mathcal{C}$ and a unique $f \in \mathcal{C}(b \to c)$ such that $U(f) = g$. In particular, $U(b) = d$.

Denote the 0-truncation of $\mathsf{hoFib}_d(U)$ by $\tau_0(\mathsf{hoFib}_d(U))$. Recall that elements of $\tau_0(\mathsf{hoFib}_d(U))$ are equivalence classes $[(c, g)]$ where $(c_1, g_1) \sim (c_2, g_2)$ if there exists an isomorphism $f : c_1 \to c_2$ such that $g_2 \circ U(f) = g_1$. Observe that if $(c, g) \in \mathsf{hoFib}_d$ and $b \in \mathcal{C}$ such that $U(b) = d$ and $f : b \to c$ such that $U(f) = g$, then $[(b, \mathrm{id}_b)] = [(c, g)]$.

Define $\Phi : \tau_0(\mathsf{hoFib}_d(U)) \to \mathsf{stFib}(U)$ by $[(c, g)] \mapsto b$ where $U(b) = d$ and $U(f) = g$. It is straightforward to verify that this map is well defined. Going the other way, define $\Psi : \mathsf{stFib}_d(U) \to \tau_0(\mathsf{hoFib}_d(U))$ by $b \mapsto [(b, \mathrm{id}_d)]$. It is straightforward to show these maps are mutually inverse. ■

Example 3.8. Let $\mathcal{C}$ be a fusion category and $Z(\mathcal{C})$ its Drinfeld center. Consider the forgetful tensor functor $\mathrm{Forget}_Z : Z(\mathcal{C}) \to \mathcal{C}$, which is faithful. Its restriction to cores $\mathrm{Forget}_Z : \mathrm{core}(Z(\mathcal{C})) \to \mathrm{core}(\mathcal{C})$ is also a discrete fibration. Given a $c \in \mathcal{C}$, the 0-truncation of the homotopy fiber $\tau_0(\mathsf{hoFib}_c(\mathrm{Forget}_Z))$ is in canonical bijection with the strict fiber $\mathsf{stFib}_c(\mathrm{Forget}_Z)$, which we view as the *set of half-braidings on c*.

Example 3.9. Suppose $\mathcal{C} \subset \mathcal{D}$ is a (fully faithful) inclusion of fusion categories. The forgetful functor $\mathrm{Forget}_Z : Z_{\mathcal{C}}(\mathcal{D}) \to \mathcal{D}$ from the relative Drinfeld center to $\mathcal{D}$ is fully faithful. Moreover, its restriction to cores $\mathrm{Forget}_Z : \mathrm{core}(Z_{\mathcal{C}}(\mathcal{D})) \to \mathrm{core}(\mathcal{D})$ is also a discrete fibration. Given a $d \in \mathcal{D}$, the 0-truncation of the homotopy fiber $\tau_0(\mathsf{hoFib}_d(\mathrm{Forget}_Z))$ is in canonical bijection with the strict fiber $\mathsf{stFib}_d(\mathrm{Forget}_Z)$, which we view as the *set of relative half-braidings on d with $\mathcal{C}$*.

Example 3.10. Let $\mathcal{C}$ be a fusion category and $(\rho, \rho^1, \rho^2) : \underline{G} \to \underline{\mathrm{Aut}_\otimes}(\mathcal{C})$ a categorical G-action on $\mathcal{C}$ by tensor automorphisms. We write ρ_g^1, ρ_g^2 for the unitor and tensorator of ρ_g. Recall from [18, Section 2.7] that the *equivariantization $\mathcal{C}^G$* has

- objects are $c \in \mathcal{C}$ together with a family of isomorphisms $\lambda_g \in \mathcal{C}(g(c) \to c)$ for $g \in G$ such that the following diagram commutes:

$$\begin{tikzcd} \rho_g(\rho_h(c)) \arrow[r, "\rho_g(\lambda_h)"] \arrow[d, "(\rho^2_{g,h})_c"] & \rho_g(c) \arrow[d, "\lambda_g"] \\ \rho_{gh}(c) \arrow[r, "\lambda_{gh}"] & c; \end{tikzcd}$$

- morphisms are $f \in \mathcal{C}((c, \lambda_g) \to (d, \kappa_g))$ such that the following diagram commutes for all $g \in G$:

$$\begin{tikzcd} \rho_g(c) \arrow[r, "g(f)"] \arrow[d, "\lambda_g"] & \rho_g(d) \arrow[d, "\kappa_g"] \\ c \arrow[r, "f"] & c. \end{tikzcd}$$

The equivariantization tensor product given by

$$(c, \lambda_g) \otimes (d, \kappa_g) := (c \otimes d, (\lambda_g \otimes \kappa_g) \circ (\rho_g^2)^{-1}_{c,d})$$

and unit object $(1_{\mathcal{C}}, \mathrm{id}_{1_{\mathcal{C}}})$.

We have an obvious faithful forgetful tensor functor $\mathrm{Forget}_G : \mathcal{C}^G \to \mathcal{C}$, which forgets the G-equivariant structure. Its restriction to cores $\mathrm{core}(\mathcal{C}^G) \to \mathrm{core}(\mathcal{C})$ is also a discrete fibration. Given $c \in \mathcal{C}$, the 0-truncation of the homotopy fiber $\tau_0(\mathsf{hoFib}_c(\mathrm{Forget}_G))$ is in canonical bijection with the strict fiber $\mathsf{stFib}_c(\mathrm{Forget}_G)$, which we view as the *set of G-equivariant structures on c*.

Example 3.11 (Set of lifts of a (monoidal) functor). Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are (monoidal) categories and $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ are (monoidal) functors. The *category of lifts of F* is the homotopy fiber $\mathsf{hoFib}_F(G \circ -)$ where $G \circ - : \mathrm{core}(\mathrm{Hom}(\mathcal{A} \to \mathcal{B})) \to \mathrm{core}(\mathrm{Hom}(\mathcal{A} \to \mathcal{C}))$. In more detail,

- objects are pairs $(\widetilde{F}, \alpha)$ with $\widetilde{F} : \mathcal{A} \to \mathcal{B}$ a (monoidal) functor and $\alpha : F \Rightarrow G \circ \widetilde{F}$ a (monoidal) equivalence and
- 1-morphisms $(\widetilde{F}_1, \alpha_1) \to (\widetilde{F}_2, \alpha_2)$ are (monoidal) natural isomorphisms $\eta : \widetilde{F}_1 \Rightarrow \widetilde{F}_2$ such that

$$\begin{tikzcd} & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[ur, "\widetilde{F}_1"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3ex, "\alpha_1 \Downarrow"] & \mathcal{C} \end{tikzcd} \quad = \quad \begin{tikzcd} & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[ur, bend left=60, "\widetilde{F}_1"] \arrow[ur, "\widetilde{F}_2"'] \arrow[ur, phantom, shift left=3ex, "\eta \Downarrow"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3ex, "\alpha_2 \Downarrow"] & \mathcal{C}. \end{tikzcd}$$

When G is 0-truncated, $G \circ -$ is 0-truncated. If moreover the restriction $G : \mathrm{core}(\mathcal{B}) \to \mathrm{core}(\mathcal{C})$ is a discrete fibration, then so is the restriction $G \circ - : \mathrm{core}(\mathrm{Hom}(\mathcal{A} \to \mathcal{B})) \to \mathrm{core}(\mathrm{Hom}(\mathcal{A} \to \mathcal{C}))$. We leave the verification of this enjoyable exercise to the reader. In this case, the 0-truncation of the homotopy fiber $\tau_0(\mathsf{hoFib}_F(G \circ -)$ is in canonical bijection with the strict fiber $\mathsf{stFib}_F(G \circ -)$, which we view as the *set of lifts of* F. These are exactly the functors $\widetilde{F} : \mathcal{A} \to \mathcal{B}$ such that $G \circ \widetilde{F} = F$ *on the nose*.

3.2.2 $n = 2$

Suppose $\mathcal{C}, \mathcal{D}$ are 2-groupoids and $U : \mathcal{C} \to \mathcal{D}$ is a 0-truncated 2-functor, that is, fully faithful on 2-morphisms. Again, in order to strictify the homotopy fiber, it is necessary, but not sufficient, that U is a fibration in the canonical model structure on bicategories [32, Section 2].

Definition 3.12. By a slight abuse of notation, we call such a 0-truncated 2-functor $U : \mathcal{C} \to \mathcal{D}$ a *discrete fibration* if for each $c \in \mathcal{C}$ and 1-morphism $g \in \mathcal{D}(d \to U(c))$, there is a unique $b \in \mathcal{C}$ and a unique 1-morphism $f \in \mathcal{C}(b \to c)$ such that $U(b) = d$ and $U(f) = g$.

Definition 3.13. We define the *strict fiber* $\mathsf{stFib}_d(U)$ of U at d is the set of objects $c \in \mathcal{C}$ such that $U(c) = d$.

The proof of the following proposition is similar to Proposition 3.7 and omitted.

Proposition 3.14. Suppose $U : \mathcal{C} \to \mathcal{D}$ is a 0-truncated 2-functor of small 2-groupoids, which is a discrete fibration. For every $d \in \mathcal{D}$, there is a canonical bijection $\mathsf{stFib}_d(U) \cong \tau_0(\mathsf{hoFib}_d(U))$.

Example 3.15 (Set of braidings). The strict 2-category $\mathsf{BrdMonCat}$ of braided monoidal categories, braided monoidal functors, and monoidal natural transformations admits a strict forgetful 2-functor $\mathrm{Forget}_{\mathrm{br}}$ to the strict 2-category MonCat of monoidal categories, monoidal functors, and monoidal natural transformations, which is fully faithful on 2-morphisms, since every monoidal natural transformation of braided monoidal functors is compatible with the braidings. Moreover, its restriction to cores is a discrete fibration; indeed, if $\mathcal{B}$ is a braided monoidal category, $\mathcal{C}$ is a monoidal category, and $F : \mathcal{C} \to \mathrm{Forget}_{\mathrm{br}}(\mathcal{B})$ is any monoidal equivalence, there is a unique way to transport the braiding on $\mathcal{B}$ to a braiding on $\mathcal{C}$ such that F is a braided equivalence. Fixing a monoidal category $\mathcal{C} \in \mathrm{core}(\mathsf{MonCat})$, the homotopy fiber $\mathsf{hoFib}_{\mathcal{C}}(\mathrm{Forget}_{\mathrm{br}})$ is 0-truncated, that is, a set. Moreover, its 0-truncation $\tau_0(\mathsf{hoFib}_{\mathcal{C}}(\mathrm{Forget}_{\mathrm{br}}))$ is in canonical bijection with the strict fiber $\mathsf{stFib}_{\mathcal{C}}(\mathrm{Forget}_{\mathrm{br}})$, which we view as the *set of braidings* on $\mathcal{C}$.

Example 3.16 (Set of G-gradings). The strict 2-category $G\mathsf{GrdMonCat}$ of G-graded monoidal categories, G-graded monoidal functors, and natural transformations admits a strict forgetful 2-functor Forget_G to the strict 2-category MonCat, which is fully faithful on 2-morphisms, since every monoidal natural transformation of G-graded monoidal functors is compatible with the gradings. Moreover, its restriction to cores $\mathrm{Forget}_G : \mathrm{core}(G\mathsf{GrdMonCat}) \to \mathrm{core}(\mathsf{MonCat})$ is a discrete fibration. Fixing a monoidal category $\mathcal{C}$, the 0-truncation of the homotopy fiber $\tau_0(\mathsf{hoFib}_{\mathcal{C}}(\mathrm{Forget}_G))$ is in canonical bijection with the strict fiber $\mathsf{stFib}_{\mathcal{C}}(\mathrm{Forget}_G)$. We think of this set as the *set of G-gradings* on $\mathcal{C}$.

Example 3.17 (Equivalence classes of G-crossed braidings). The strict 2-category $G\mathsf{CrsBrd}$ of G-crossed braided fusion categories, G-crossed braided monoidal functors, and natural transformations admits a strict forgetful 2-functor Forget_β to the strict 2-groupoid $G\mathsf{GrdFusCat}$ of G-graded fusion categories, G-graded monoidal functors, and natural transformations that is fully faithful on 2-morphisms.

Unfortunately, the restriction to cores $\mathrm{Forget}_\beta : \mathrm{core}(G\mathsf{CrsBrd}) \to \mathrm{core}(G\mathsf{GrdFusCat})$ is *not* a discrete fibration, as given a G-crossed braided fusion category $\mathcal{C}$ and a G-graded equivalence to another G-graded fusion category $\mathcal{D}$, there is not a unique G-crossed braiding on $\mathcal{D}$ such that $\mathcal{C}, \mathcal{D}$ are G-crossed braided equivalent. This discrepancy arises as there is not a unique G-action on $\mathcal{D}$ compatible with the equivalence $\mathcal{C} \cong \mathcal{D}$, but an equivalence class.

Suppose $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ is a G-graded fusion category and $(\rho^1, \beta^1), (\rho^2, \beta^2)$ are two G-crossed braidings on $\mathcal{D}$. We say $(\rho^1, \beta^1), (\rho^2, \beta^2)$ are *equivalent* if there is an equivalence $\eta : \rho^1 \Rightarrow \rho^2$ of monoidal functors $\underline{G} \to \underline{\mathrm{Aut}}^{\otimes}(\mathcal{D})$ such that for all $x_g \in \mathcal{D}_g$ and $y \in \mathcal{D}$,

$$(\eta^g_y \otimes \mathrm{id}_{x_g}) \circ \beta^1_{x_g, y} = \beta^2_{x_g, y} : x_g \otimes y \to \rho^2_g(y) \otimes x_g.$$

Observe that there is *at most one* equivalence between any two G-crossed braidings as the monoidal natural isomorphism η is completely determined by β^1, β^2 if it exists. (Indeed, $\beta^1_{x_g, y}$ is invertible and $-\otimes \mathrm{id}_c$ is injective on hom spaces for every fusion category using [25, Lemma A.5].)

Thus, fixing a G-graded fusion category $\mathcal{D}$, the 0-truncation of the homotopy fiber $\tau_0(\mathsf{hoFib}_{\mathcal{D}}(\mathrm{Forget}_\beta))$ of $\mathrm{Forget}_\beta : \mathrm{core}(G\mathsf{CrsBrd}) \to \mathrm{core}(G\mathsf{GrdFusCat})$ is in canonical bijection with the set of equivalence classes G-crossed braidings on $\mathcal{D}$.

Example 3.18 (Set of lifts of a (monoidal) 2-functor). Given two (monoidal) 2-categories $\mathcal{A}, \mathcal{B}$, there is a 2-category $\mathrm{Hom}(\mathcal{A} \to \mathcal{B})$ whose objects are (monoidal) 2-functors, 1-morphisms are (monoidal) natural transformations, and 2-morphisms are (monoidal) modifications. Now, suppose $G : \mathcal{B} \to \mathcal{C}$ is a (monoidal) 2-functor between (monoidal) 2-categories and $\mathcal{A}$ is another (monoidal) 2-category. We have a 2-functor $G \circ - : \mathrm{Hom}(\mathcal{A} \to \mathcal{B}) \to \mathrm{Hom}(\mathcal{A} \to \mathcal{C})$. As in Example 3.11, if G is 0-truncated, then so is $G \circ -$. Moreover, if G is a discrete fibration, then so is $G \circ -$. Fixing $F \in \mathrm{Hom}(\mathcal{A} \to \mathcal{C})$, the 0-truncation of the homotopy fiber $\tau_0(\mathsf{hoFib}_F(G \circ -))$ is in canonical bijection with the strict fiber $\mathsf{stFib}_F(G \circ -)$ of (monoidal) 2-functors $\widetilde{F} : \mathcal{A} \to \mathcal{B}$ such that $G \circ \widetilde{F} = F$ *on the nose*. We view this set as the *set of lifts* of F.

3.3 G-Gradings on fusion categories

Fix a finite group G. In this section, we explain how G-gradings on a fixed fusion category $\mathcal{T}$ may be characterized in terms of braided enriched structure. In this section, our fusion categories are always over an algebraically closed field $\Bbbk$ of characteristic zero. (Although it is not necessary, we would even be happy to assume further that $\Bbbk = \mathbb{C}$.)

Definition 3.19. Suppose G is a group and $\mathcal{T} = \bigoplus_{g \in G} \mathcal{T}_g$ is a faithfully G-graded fusion category. In this case, by [21, p. 12], there is a canonical fully faithful strong monoidal functor $\mathcal{I} = \mathcal{I}^{\mathcal{T}} : \mathsf{Rep}(G) \to Z(\mathcal{T})$ defined as follows. For a representation $(K, \pi) \in \mathsf{Rep}(G)$, we consider the object $\mathcal{I}_\pi := K \otimes 1_{\mathcal{T}} \in \mathcal{T}$. Notice that both $\mathcal{I}_\pi \otimes t$ and $t \otimes \mathcal{I}_\pi$ are canonically isomorphic to $K \otimes t$. Thus, we can endow $\mathcal{I}_\pi$ with the half-braiding

$$\zeta_{t, \mathcal{I}_\pi} := \pi_g \otimes \mathrm{id}_t : t \otimes \mathcal{I}_\pi \cong K \otimes t \longrightarrow K \otimes t \cong \mathcal{I}_\pi \otimes t \qquad t \in \mathcal{T}_g.$$

For a morphism $f : (K, \pi) \to (L, \rho)$, we get a morphism $\mathcal{I}_f := f \otimes \mathrm{id}_{1_{\mathcal{T}}} : \mathcal{I}_\pi \to \mathcal{I}_\rho$. It is straightforward to verify

- $\mathcal{I}$ is a fully faithful strong monoidal functor (using the obvious tensorator/strength) since $\mathcal{T}$ is tensored over Vec;
- the forgetful functor $\mathrm{Forget}_Z : Z(\mathcal{T}) \to \mathcal{T}$ restricted to this copy of $\mathsf{Rep}(G) \subseteq Z(\mathcal{T})$ is canonically monoidally naturally isomorphic to the canonical symmetric monoidal fiber functor

$$\mathrm{Forget}_{\mathsf{Rep}} : \mathsf{Rep}(G) \to \mathsf{Vec} \cong \langle 1_{\mathcal{T}}, \mathrm{id}_{1_{\mathcal{T}}} \rangle \subseteq \mathcal{T}.$$

The following important lemma is essentially in [21]. Recall that given a braided fusion category $(\mathcal{V}, \beta)$ and a symmetric subcategory $\mathcal{S} \subset \mathcal{V}$, the *Müger centralizer* $\mathcal{S}'$ of $\mathcal{S} \subset \mathcal{V}$ is the full subcategory of $\mathcal{V}$ whose objects are *transparent* to $\mathcal{S}$, that is, $\beta_{v,s} \circ \beta_{s,v} = \mathrm{id}_{s \otimes v}$ for all $s \in \mathcal{S}$.

Lemma 3.20. An object $(t, \sigma_{\bullet,t}) \in \mathsf{Rep}(G)' \subseteq Z(\mathcal{T})$ if and only if $\mathrm{Forget}_Z(t, \sigma_{\bullet,t}) = t \in \mathcal{T}_e$.

Proof. It is clear that $t \in \mathcal{T}_e$ implies $(t, \sigma_{\bullet,t}) \in \mathsf{Rep}(G)'$. Suppose $(t, \sigma_{\bullet,t}) \in \mathsf{Rep}(G)'$. If $t = \bigoplus_g t_g$, then for all $(H, \pi) \in \mathsf{Rep}(G)$,

$$\mathrm{id}_{\mathcal{I}_\pi \otimes t} = \zeta_{t,\mathcal{I}_\pi} \circ \sigma_{\mathrm{Forget}_Z(\mathcal{I}_\pi),t} = \zeta_{t,\mathcal{I}_\pi} \circ \sigma_{\bigoplus 1_{\mathcal{T}},t} = \bigoplus_g (\pi_g \otimes \mathrm{id}_{t_g}).$$

Since $\mathcal{T}$ is fusion, the above holds if and only if $t_g = 0$ for all $g \neq e$. ■

Definition 3.21. Suppose $(\mathcal{V}, F : \mathcal{V} \to \mathsf{Vec})$ is a braided fusion category equipped with a fixed faithful strong monoidal fiber functor. Given a fixed fusion category $\mathcal{T}$, a $\mathcal{V}$-*fibered enrichment* of $\mathcal{T}$ is a braided strong monoidal functor $\mathcal{F}^Z : \mathcal{V} \to Z(\mathcal{T})$ together with a monoidal natural isomorphism

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\mathcal{F}^Z"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\alpha}{\Nearrow}"] & Z(\mathcal{T}) \arrow[d, "\mathrm{Forget}_Z"] \\ \mathsf{Vec} \arrow[r, hook, "i_{\mathcal{T}}"] & \mathcal{T} \end{tikzcd} \qquad \alpha : \mathcal{F} := \mathrm{Forget}_Z \circ \mathcal{F}^Z \Rightarrow i_{\mathcal{T}} \circ F,$$

where $i_{\mathcal{T}} : \mathsf{Vec} = \langle 1_{\mathcal{T}} \rangle \hookrightarrow \mathcal{T}$ is the inclusion $V \mapsto V \otimes 1_{\mathcal{T}}$.

Similar to $\mathcal{V}$-module tensor categories as in Definition 2.6, $\mathcal{V}$-fibered enrichments form a 2-category $\mathcal{V}\mathsf{FibFusCat}$. A 1-morphism $(\mathcal{T}_1, \mathcal{F}_1^Z, \alpha^1) \to (\mathcal{T}_2, \mathcal{F}_2^Z, \alpha^2)$ is a 1-morphism $(H, \eta) : (\mathcal{T}_1, \mathcal{F}_1^Z) \to (\mathcal{T}_2, \mathcal{F}_2^Z)$ of the underlying $\mathcal{V}$-module tensor categories satisfying the extra compatibility with α^1, α^2:

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\mathcal{F}_1^Z"] \arrow[d, "F"'] \arrow[rr, bend left=50, "\mathcal{F}_2^Z", ""{name=U}] \arrow[dr, phantom, "\overset{\alpha^1}{\Nearrow}"] & Z(\mathcal{T}_1) \arrow[d, "\mathrm{Forget}_Z"] \arrow[phantom, u, "\Downarrow\eta"] & Z(\mathcal{T}_2) \arrow[d, "\mathrm{Forget}_Z"] \\ \mathsf{Vec} \arrow[r, hook, "i_{\mathcal{T}_1}"] \arrow[rr, hook, bend right=50, "i_{\mathcal{T}_2}"'] & \mathcal{T}_1 \arrow[r, "H"] \arrow[phantom, d, "H^1\Uparrow"] & \mathcal{T}_2 \\ & {} & \end{tikzcd} = \begin{tikzcd} \mathcal{V} \arrow[r, "\mathcal{F}_2^Z"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\alpha^2}{\Nearrow}"] & Z(\mathcal{T}_2) \arrow[d, "\mathrm{Forget}_Z"] \\ \mathsf{Vec} \arrow[r, hook, "i_{\mathcal{T}_2}"] & \mathcal{T}_2 \end{tikzcd} . \tag{3.1}$$

Above, note that the unitality constraint $H^1 : 1_{\mathcal{T}_2} \to G(1_{\mathcal{T}_1})$ determines a monoidal natural isomorphism still denoted $H^1 : i_{\mathcal{T}_2} \Rightarrow H \circ i_{\mathcal{T}_1}$ of monoidal functors $\mathsf{Vec} \to \mathcal{T}_2$. Moreover, observe that η is completely determined, as α^1, α^2, H^1 are all isomorphisms; indeed, (3.1) above is equivalent to, which is equivalent to

$$\eta_v = H(\alpha^1_v) \circ (H^1)^{-1}_{F(v)} \circ (\alpha^2_v)^{-1} \qquad \forall\, v \in \mathcal{V}. \tag{3.2}$$

A 2-morphism $\kappa : (H_1, \eta^1) \Rightarrow (H_2, \eta^2)$ is an *arbitrary* monoidal natural transformation $\kappa : H_1 \Rightarrow H_2$. Observe that the extra compatibility with the fiber functor F amounts to unitality of the monoidal natural isomorphism κ, and the extra action coherence (2.2) with η^1, η^2 is automatically satisfied. Indeed, setting $\mathcal{F}_i := \mathrm{Forget}_Z \circ \mathcal{F}_i^Z$ for $i = 1, 2$, (2.2) automatically commutes by naturality and unitality of κ:

$$\begin{tikzcd}
\mathcal{F}_2(v) \arrow[r, "(\alpha^2_v)^{-1}"] \arrow[d, equal] \arrow[rrr, bend left, "\eta^1_v", "= \text{by (3.2)}"'] & F(v) \otimes 1_{\mathcal{T}_2} \arrow[r, "(H^1_1)^{-1}_{F(v)}"] \arrow[d, equal] & H_1(F(v) \otimes 1_{\mathcal{T}_1}) \arrow[r, "H_1(\alpha^1_v)"] \arrow[d, Rightarrow, "\kappa_{F(v)\otimes 1_{\mathcal{T}_1}}"] & H_1(\mathcal{F}_1(v)) \arrow[d, Rightarrow, "\kappa_{\mathcal{F}_1(v)}"] \\
\mathcal{F}_2(v) \arrow[r, "(\alpha^2_v)^{-1}"] \arrow[rrr, bend right, "\eta^2_v"', "= \text{by (3.2)}"] & F(v) \otimes 1_{\mathcal{T}_2} \arrow[r, "(H^1_2)^{-1}_{F(v)}"] & H_2(F(v) \otimes 1_{\mathcal{T}_1}) \arrow[r, "H_2(\alpha^1_v)"] & H_2(\mathcal{F}_1(v))
\end{tikzcd}$$

The following lemma allows us to work with an equivalent strict $\mathcal{V}$-fibered enrichment.

Lemma 3.22. The 2-category $\mathcal{V}\mathsf{FibFusCat}$ is equivalent to the full 2-subcategory $\mathcal{V}\mathsf{FibFusCat}^{\mathsf{st}}$ with objects $(\mathcal{T}, \mathcal{F}^Z, \mathrm{id})$, that is, $\mathrm{Forget}_Z \circ \mathcal{F}^Z = i_{\mathcal{T}} \circ F$ on the nose.

Proof. Suppose $(\mathcal{T}, \mathcal{F}^Z, \alpha) \in \mathcal{V}\mathsf{FibFusCat}$. Define $(\mathcal{T}, \mathcal{F}'^Z, \mathrm{id}) \in \mathcal{V}\mathsf{FibFusCat}$ by $\mathcal{F}'(v) := F(v) \otimes 1_{\mathcal{T}}$ and $\sigma'_{t,\mathcal{F}'(v)} := (\alpha_v^{-1} \otimes \mathrm{id}_t) \circ \sigma_{\mathcal{F}(z),t} \circ (\mathrm{id}_t \otimes \alpha_v)$ with tensorator $\mu'_{u,v} := \alpha^{-1}_{u \otimes v} \circ \mu_{u,v} \circ (\alpha_u \otimes \alpha_v)$. By definition, we have $\mathcal{F}' := \mathrm{Forget}_Z \circ \mathcal{F}'^Z = i_{\mathcal{T}} \circ F$ on the nose, so $(\mathcal{T}, \mathcal{F}'^Z, \mathrm{id}) \in \mathcal{V}\mathsf{FibFusCat}^{\mathsf{st}}$. We claim that $(H := \mathrm{id}_{\mathcal{T}}, \eta := \alpha) : (\mathcal{T}, \mathcal{F}^Z, \alpha) \to (\mathcal{T}, \mathcal{F}'^Z, \mathrm{id})$ defines an invertible 1-morphism in $\mathcal{V}\mathsf{FibFusCat}$. It is clear that $\alpha^1 = \alpha, \alpha^2 = \mathrm{id}, \eta = \alpha$ satisfies (3.2). It remains to check that $\eta = \alpha$ satisfies the half-braiding coherence (2.1), which expands to the formula $(\alpha_v \otimes \mathrm{id}_t) \circ \sigma'_{t,\mathcal{F}'(v)} = \sigma_{t,\mathcal{F}(z)} \circ (\mathrm{id}_t \otimes \alpha_v)$, which holds by definition. ■

The following theorem shows that fully faithful $\mathsf{Rep}(G)$-fibered enriched fusion categories are the same as faithfully G-graded fusion categories. We denote by $G\mathsf{GrdFusCat}_{\mathsf{f}}$ the 2-category of *faithfully* G-graded fusion categories and by $\mathsf{Rep}(G)\mathsf{FibFusCat}_{\mathsf{ff}}$ the 2-category of *fully faithful* $\mathsf{Rep}(G)$-fibered enriched fusion categories, where we endow $\mathsf{Rep}(G)$ with the canonical symmetric fiber functor to Vec.

Theorem 3.23. There is a strict 2-equivalence $\Phi : G\mathsf{GrdFusCat}_{\mathsf{f}} \to \mathsf{Rep}(G)\mathsf{FibFusCat}_{\mathsf{ff}}$ such that the following triangle commutes:

$$\begin{tikzcd} G\mathsf{GrdFusCat}_{\mathsf{f}} \arrow[rr, "\Phi"] \arrow[dr, "\mathrm{Forget}_G"'] & & \mathsf{Rep}(G)\mathsf{FibFusCat}_{\mathsf{ff}} \arrow[dl, "\mathrm{Forget}_{\mathsf{Rep}(G)}"] \\ & \mathsf{FusCat} & \end{tikzcd} \tag{3.3}$$

Here, $\mathrm{Forget}_G : G\mathsf{GrdFusCat} \to \mathsf{FusCat}$ forgets the G-grading (cf. Example 3.16) and $\mathrm{Forget}_{\mathsf{Rep}(G)} : \mathcal{V}\mathsf{FibFusCat} \to \mathsf{FusCat}$ forgets the $\mathsf{Rep}(G)$-fibered enrichment.

Proof. We already saw in Definition 3.19 how to endow a faithfully G-graded fusion category $\mathcal{T}$ with a fully faithful $\mathsf{Rep}(G)$-fibered enrichment. It is straightforward to show that every G-graded monoidal functor $H : \mathcal{T}_1 \to \mathcal{T}_2$ gives a 1-morphism in $\mathsf{Rep}(G)\mathsf{FibFusCat}$, since the coherence η is completely determined by (3.1). Indeed, the verification that this determined η satisfies the compatibility (2.1) with the half-braidings amounts to the following commuting square for a homogeneous $t_g \in (\mathcal{T}_1)_g$ and $(K, \pi) \in \mathsf{Rep}(G)$:

$$\begin{tikzcd} K \otimes H(t_g) \arrow[r, "\cong"] \arrow[d, "\pi_g \otimes \mathrm{id}_{H(t_g)}"] & H(K \otimes t_g) \arrow[d, "H(\pi_g \otimes \mathrm{id}_{t_g})"] \\ K \otimes H(t_g) \arrow[r, "\cong"] & H(K \otimes t_g) \end{tikzcd} .$$

Moreover, the 2-morphisms of both $G\mathsf{GrdFusCat}_{\mathsf{f}}$ and $\mathsf{Rep}(G)\mathsf{FibFusCat}_{\mathsf{ff}}$ consist of *all* monoidal natural transformations, so Φ is the identity on 2-morphisms. We leave it to the reader to check that Φ is a strict 2-functor, which is obviously fully faithful on 2-morphisms such that (3.3) commutes.

It remains to show essential surjectivity on objects and 1-morphisms. By Lemma 3.22, we may restrict our attention to the 2-subcategory $\mathsf{Rep}(G)\mathsf{FibFusCat}^{\mathsf{st}}_{\mathsf{ff}}$ of fully faithful $\mathsf{Rep}(G)$-fibered enriched fusion categories $(\mathcal{T}, \mathcal{I}^Z)$ such that $\mathrm{Forget}_Z \circ \mathcal{I}^Z = i_{\mathcal{T}} \circ F$ on the nose.

Given $(\mathcal{T}, \mathcal{I}^z) \in \mathsf{Rep}(G)\mathsf{FibFusCat}^{\mathsf{st}}_{\mathsf{ff}}$, we claim there is a canonical faithful G-grading on $\mathcal{T}$ that recovers our $\mathsf{Rep}(G)$-fibered enrichment. We expect this result is known to experts, but we are unaware of its existence in the literature.

Recall $\mathcal{O}(G)$ is the commutative algebra of $\Bbbk$-valued functions on G. Moreover, $\mathcal{O}(G)$ is a Hopf algebra with comultiplication given by $\Delta(\chi_g) := \sum_h \chi_{gh^{-1}} \otimes \chi_h$ where χ_g denotes the indicator function at $g \in G$, antipode given by $S\chi_g := \chi_{g^{-1}}$, and counit given by $\epsilon(\chi_g) = \delta_{g=e}$. Let $\operatorname{Irr}(\mathsf{Rep}(G))$ be a set of representatives for the simple objects of $\mathsf{Rep}(G)$. There is a unital isomorphism of Hopf algebras

$$\Phi : \bigoplus_{(K,\pi)\in\operatorname{Irr}(\mathsf{Rep}(G))} (K,\pi)^* \otimes (K,\pi) \cong \mathcal{O}(G) \tag{3.4}$$

given on $w^* \otimes v \in (K,\pi)^* \otimes (K,\pi)$ by $\Phi(w^* \otimes v)(g) := w^*(\pi_g(v))$. Multiplication on the left-hand side is given on $w_i^* \otimes v_i \in K_i^* \otimes K_i$ for $i = 1, 2$ by

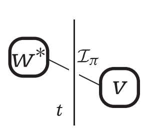
$$(w_1^* \otimes v_1)(w_2^* \otimes v_2*) = \sum_{\substack{(L,\pi)\in\operatorname{Irr}(\mathsf{Rep}(G))\\ \{\alpha\}\subset\mathsf{Rep}(G)(K_1\otimes K_2\to L)}} [(w_1^* \otimes w_2^*) \circ \alpha^*] \otimes [\alpha \circ (v_1 \otimes v_2)],$$

where $\{\alpha\} \subseteq \mathsf{Rep}(G)(K_1 \otimes K_2 \to L)$ is a basis and $\{\alpha^*\} \subset \mathsf{Rep}(G)(L \to K_1 \otimes K_2)$ is the dual basis under the pairing $\alpha' \circ \alpha^* = \delta_{\alpha'=\alpha}\mathrm{id}_L$. The unit on the left-hand side is exactly $1_{\mathbb{C}}^* \otimes 1_{\mathbb{C}} \in \mathbb{C}^* \otimes \mathbb{C}$ where $\mathbb{C} \in \mathsf{Rep}(G)$ is the trivial representation. Comultiplication on $w^* \otimes v \in K^* \otimes K$ is given by

$$\Delta(w^* \otimes v) = \sum_i (w^* \otimes e_i) \otimes (e_i^* \otimes v),$$

where $\{e_i\}$ is a basis for K and $\{e_i^*\}$ is the dual basis. We will identify both sides of (3.4) under the isomorphism Φ below.

Now, given $t \in \mathcal{T}$, we get a unital $\Bbbk$-algebra homomorphism $\mathcal{O}(G) \to \mathcal{T}(t \to t)$ (whose image lies in $Z(\mathcal{T}(t \to t))$) whose image on $w^* \otimes v \in K^* \otimes K$ is given by

$$w^* \otimes v \longmapsto$$  $$:= (w^* \otimes \mathrm{id}_t) \circ \zeta_{I_\pi,t} \circ (\mathrm{id}_t \otimes v), \tag{3.5}$$

where we identify elements $v \in K$ as morphisms $v : \Bbbk \to K$, which gives a map $v \in \mathcal{T}(1_{\mathcal{T}} \to 1_{\mathcal{T}} \otimes K = \mathcal{I}_\pi)$, and similarly $w^* \in \mathcal{T}(\mathcal{I}_\pi = 1_{\mathcal{T}} \otimes K \to 1_{\mathcal{T}})$. Now, $\mathcal{O}(G) \cong \Bbbk^{|G|}$

is an abelian $\Bbbk$-algebra, so for each $t \in \mathcal{T}$ and $g \in G$, we have a canonical projector $\chi_g^t \in \mathcal{T}(t \to t)$. The proof of the following lemma is straightforward.

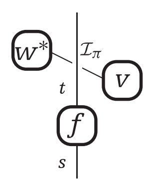
Lemma 3.24. For $t \in \mathcal{T}$, the projectors $\chi_g^t \in \mathcal{T}(t \to t)$ satisfy the relations

- (direct sum) $\chi_g^t \circ \chi_h^t = \delta_{g=h}\chi_g^t$ and $\sum_{g\in G} \chi_g^t = \mathrm{id}_t$ and
- (compatibility with morphisms) for all $s \in \mathcal{T}$ with projectors $\chi_g^s \in \mathcal{T}(s \to s)$ and all morphisms $f \in \mathcal{T}(s \to t)$, we have $\chi_g^s \circ f = f \circ \chi_g^t$. ■

As $\mathcal{T}$ is fusion and thus idempotent complete, for $g \in G$, we may define $t_g := \mathrm{im}(\chi_g^t)$. By the direct sum relation in Lemma 3.24, we have $t = \bigoplus_{g\in G} t_g$. Moreover, for all $f \in \mathcal{T}(s \to t)$, we see that $\mathcal{T}(s \to t) = \bigoplus_{g\in G} \mathcal{T}(s_g \to t_g)$. Thus, defining $\mathcal{T}_g$ to be the subcategory whose objects are of the form t_g for $t \in \mathcal{T}$, we have $\mathcal{T} = \bigoplus_{g\in G} \mathcal{T}_g$, that is, $\mathcal{T}$ is G-graded as a semisimple category.

We now claim that this G-grading is compatible with the tensor product, that is, if $s \in \mathcal{T}_g$ and $t \in \mathcal{T}_h$, then $s \otimes t \in \mathcal{T}_{gh}$. To show this, we observe that the map (3.5) endows each hom space $\mathcal{T}(s \to t)$ with an $\mathcal{O}(G)$-*action*

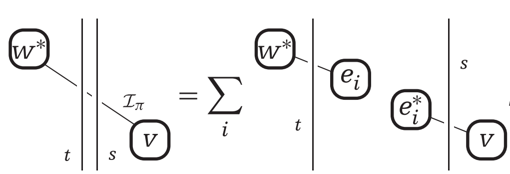
$(w^* \otimes v) \triangleright f :=$



such that

$$(w_1^* \otimes v_1)(w_2^* \otimes v_2) \triangleright f = (w_1^* \otimes v_1) \triangleright (w_2^* \otimes v_2) \triangleright f \qquad \forall f \in \mathcal{T}(s \to t). \tag{3.6}$$

Since for all $s, t \in \mathcal{T}$,



our $\mathcal{O}(G)$-action satisfies

$$(-\otimes_{\mathcal{T}} -) \circ \Delta(w^* \otimes v) \triangleright (f_1 \otimes_{\Bbbk} f_2) = (w^* \otimes v) \triangleright (f_1 \otimes f_2) \qquad \forall f_1 \in \mathcal{T}(s_1 \to t_1),\ f_2 \in \mathcal{T}(s_2 \to t_2). \tag{3.7}$$

This immediately implies that the idempotent $\chi_g^{st} \in \mathcal{T}(st \to st)$ decomposes as

$$\chi_g^{st} = \sum_{h \in G} \chi_{gh}^{s} \otimes \chi_{h^{-1}}^{t} \qquad \Longrightarrow \qquad \chi_{gh}^{st} \circ (\chi_g^s \otimes \chi_h^t) = \chi_g^s \otimes \chi_h^t \qquad \forall g, h \in G.$$

Thus, the G-grading on $\mathcal{T}$ respects the tensor product of $\mathcal{T}$. We leave it to the reader to verify this G-grading recovers an equivalent $\mathsf{Rep}(G)$-fibered enrichment on $\mathcal{T}$.

Finally, we show essential surjectivity on 1-morphisms. Suppose $(H, \eta) : (\mathcal{T}_1, \mathcal{I}_1^Z) \to (\mathcal{T}_2, \mathcal{I}_2^Z)$ is a 1-morphism in $\mathcal{V}\mathsf{FibFusCat}_{\mathsf{ff}}^{\mathsf{st}}$. It suffices to prove that H is G-graded, since η is completely determined by H by (3.2). By using the compatibility of η with the half-braidings (2.1), we see that H intertwines the $\mathcal{O}(G)$-actions (3.5) on $\mathcal{T}_1(t \to t)$ and $\mathcal{T}_2(H(t) \to H(t))$ for all $t \in \mathcal{T}_1$. Thus, H maps $\chi_g^t \in \mathcal{T}_1(t \to t)$ to $\chi_g^{H(t)} \in \mathcal{T}_2(H(t) \to H(t))$, and H is G-graded.

We now look at the equivalent 2-subcategory $\mathsf{Rep}(G)\mathsf{FibFusCat}_{\mathsf{ff}}^{\mathsf{st}}$ such that $\mathrm{Forget}_Z \circ \mathcal{I}^Z = i_{\mathcal{T}} \circ F$ on the nose. On this 2-subcategory, the restriction to cores of the forgetful 2-functor $\mathrm{Forget}_{\mathsf{Rep}(G)}$ is a discrete fibration, since given any fully faithful $\mathsf{Rep}(G)$-fibered enriched fusion category $(\mathcal{T}_1, \mathcal{I}_1^Z)$ such that $\mathrm{Forget}_Z \circ \mathcal{I}_1^Z = i_{\mathcal{T}_1} \circ F$ and any monoidal equivalence $H : \mathcal{T}_1 \to \mathcal{T}_2$, there is a unique lift $\mathcal{I}_2^Z : \mathsf{Rep}(G) \to Z(\mathcal{T}_2)$ such that $\mathrm{Forget}_Z \circ \mathcal{I}_2^Z = i_{\mathcal{T}_2} \circ F$, which completely determines the necessary action coherence morphism η to make (H, η) an invertible 1-morphism in $\mathsf{Rep}(G)\mathsf{FibFusCat}_{\mathsf{ff}}^{\mathsf{st}}$. We conclude that the 0-truncation of the homotopy fiber $\tau_0(\mathsf{hoFib}_{\mathcal{T}}(\mathrm{Forget}_{\mathsf{Rep}(G)}))$ of the forgetful 2-functor $\mathrm{Forget}_{\mathsf{Rep}(G)} : \mathrm{core}(\mathsf{Rep}(G)\mathsf{FibFusCat}_{\mathsf{ff}}^{\mathsf{st}}) \to \mathrm{core}(\mathsf{FusCat})$ is in canonical bijection with the strict fiber $\mathsf{stFib}_{\mathcal{T}}(\mathrm{Forget}_{\mathsf{Rep}(G)})$, which we view as the *set of $\mathcal{V}$-fibered enrichments of $\mathcal{T}$*.

Corollary 3.25. Fix a fusion category $\mathcal{T}$ and a finite group G, and denote by $F : \mathsf{Rep}(G) \to \mathsf{Vec} = \langle 1_{\mathcal{T}} \rangle \subset \mathcal{T}$ the canonical symmetric fiber functor. There is a bijective correspondence between the sets of

1. fully faithful $\mathsf{Rep}(G)$-fibered enrichments $\mathcal{I}^Z : \mathsf{Rep}(G) \to Z(\mathcal{T})$ such that $\mathrm{Forget}_Z \circ \mathcal{I}^Z = i_{\mathcal{T}} \circ F$ on the nose, where $i_{\mathcal{T}} : \mathsf{Vec} \hookrightarrow \mathcal{T}$ is the canonical inclusion $V \mapsto V \otimes 1_{\mathcal{T}}$; and
2. the set of faithful G-gradings on $\mathcal{T}$ (cf. Example 3.16).

Proof. The equivalence $\Phi : G\mathsf{GrdFusCat}_{\mathsf{f}} \to \mathsf{Rep}(G)\mathsf{FibFusCat}_{\mathsf{ff}}$ such that the triangle (3.3) commutes gives an equivalence when restricted to cores such that the obvious triangle of cores commutes. This gives an equivalence of the 0-truncated homotopy fibers over $\mathcal{T}$ of Forget_G : $\mathrm{core}(G\mathsf{GrdFusCat}_{\mathsf{f}}) \to \mathrm{core}(\mathsf{FusCat})$ and $\mathrm{Forget}_{\mathsf{Rep}(G)}$: $\mathrm{core}(\mathsf{Rep}(G)\mathsf{FibFusCat}^{st}_{\mathsf{ff}}) \to \mathrm{core}(\mathsf{FusCat})$. Since both Forget_G and $\mathrm{Forget}_{\mathsf{Rep}(G)}$ restricted to cores are fully faithful and discrete fibrations (the former by Example 3.16 and the latter by the discussion before the corollary), we get canonical bijections

$$\mathsf{stFib}_{\mathcal{T}}(\mathrm{Forget}_G) \cong \tau_0(\mathsf{hoFib}_{\mathcal{T}}(\mathrm{Forget}_G)) \cong \tau_0(\mathsf{hoFib}_{\mathcal{T}}(\mathrm{Forget}_{\mathsf{Rep}(G)})) \cong \mathsf{stFib}_{\mathcal{T}}(\mathrm{Forget}_{\mathsf{Rep}(G)}).$$

This completes the proof. ■

Remark 3.26. By [18, Corollary 3.6.6] G-gradings on a fusion category are also classified by surjective group homomorphisms from the *universal grading group* U to G.

With these results in hand, we make the following definition.

Definition 3.27. A faithfully *G-graded* $\mathcal{V}$-fusion category is a $\mathcal{V}$-fusion category $(\mathcal{D}, \mathcal{F}^{Z}_{\mathcal{D}})$ such that $\mathcal{D}$ is faithfully G-graded as an ordinary fusion category, and $\mathcal{F}^{Z}_{\mathcal{D}}(\mathcal{V}) \subseteq \mathsf{Rep}(G)' \subset Z(\mathcal{D})$.

A *G-extension* of a $\mathcal{V}$-fusion category $(\mathcal{C}, \mathcal{F}^{Z}_{\mathcal{C}})$ is a faithfully G-graded $\mathcal{V}$-fusion category $(\mathcal{D}, \mathcal{F}^{Z}_{\mathcal{D}})$ together with an equivalence of $\mathcal{V}$-fusion categories $(\mathcal{C}, \mathcal{F}^{Z}_{\mathcal{C}}) \cong (\mathcal{D}_e, \mathcal{F}^{Z}_{\mathcal{D}})$ (recall $(\mathrm{Forget}_Z \circ \mathcal{F}^{Z}_{\mathcal{D}})(\mathcal{V}) \subseteq \mathcal{D}_e$ by Lemma 3.20).

We close this section with the following observation about $\mathsf{Rep}(G)$-fibered enrichments. Given a fully faithful braided tensor functor $\mathsf{Rep}(G) \to Z(\mathcal{C})$ where $\mathcal{C}$ is a fusion category, it is not necessarily the case that $\mathcal{C}$ is G-graded. For example, taking $\mathcal{C} = \mathsf{Rep}(G)$, the universal grading group of $\mathcal{C}$ is $\widehat{Z(G)}$. Note that this enrichment is as far as possible from a $\mathsf{Rep}(G)$-fibered enrichment, since postcomposing the enrichment with the forgetful functor yields an equivalence. However, $\mathsf{Rep}(G)$ is Morita equivalent to $\mathsf{Vec}(G)$, the quintessential example of a G-graded fusion category. Our next result shows this behavior is generic. The proposition below shows that any fusion category with a $\mathsf{Rep}(G)$ enrichment is Morita equivalent to a G-graded fusion category whose associated $\mathsf{Rep}(G)$ enrichment (obtained from the canonical equivalence of centers) is fibered. This can be interpreted as a partial converse to Corollary 3.25.

Proposition 3.28. Suppose $\mathcal{C}$ is a fusion category and $F : \mathsf{Rep}(G) \to Z(\mathcal{C})$ is a fully faithful tensor functor. Then, there exists a faithfully G-graded fusion category $\mathcal{D}$ that is Morita equivalent to $\mathcal{C}$ such that the associated enrichment $\mathsf{Rep}(G) \to Z(\mathcal{C}) \cong Z(\mathcal{D})$ is a $\mathsf{Rep}(G)$-fibered enrichment.

Proof. Consider the image of $\mathcal{O}(G)$ inside $Z(\mathcal{C})$, which is a connected étale algebra, which we will still denote by $\mathcal{O}(G)$. Observe that $Z(\mathcal{C})_{\mathcal{O}(G)}$ is a G-crossed braided extension of $Z(\mathcal{C})^{\text{loc}}_{\mathcal{O}(G)}$ by [18, Theorem 8.24.3]. Now, note that $Z(\mathcal{C})^{\text{loc}}_{\mathcal{O}(G)} \cong Z(\mathcal{C}_{\mathcal{O}(G)})$ by [11, Theorem 3.20] where $\mathcal{C}_{\mathcal{O}(G)}$ is a multifusion category, and every center of a multifusion category is also the center of an ordinary fusion category [11, Remark 5.2]. By [21], there is a bijective correspondence between G-extensions of fusion categories $\mathcal{F}$ and G-crossed braided extensions of $Z(\mathcal{F})$, which is established by taking the relative center. Thus, there is a G-graded fusion category $\mathcal{D}$ whose relative center with respect to its trivial component is $Z(\mathcal{C})_{\mathcal{O}(G)}$. Furthermore, by [21], $Z(\mathcal{D}) \cong (Z(\mathcal{C})_{\mathcal{O}(G)})^G \cong Z(\mathcal{C})$. Hence, $\mathcal{D}$ is Morita equivalent to $\mathcal{C}$. Since the forgetful functor $Z(\mathcal{D}) \to \mathcal{D}$ factors through $Z(\mathcal{C})_{\mathcal{O}(G)}$, the $\mathsf{Rep}(G)$-enrichment for $\mathcal{D}$ is fibered. ∎

4 Lifting $\mathcal{V}$-Enrichment to a Fixed G-Extension

For this section, we fix a braided fusion category $\mathcal{V}$, a $\mathcal{V}$-fusion category $(\mathcal{C}, \mathcal{F}^Z) \in \mathsf{BrdFus}_1(\mathcal{V} \to \mathsf{Vec})$.

Definition 4.1. A $\mathcal{V}$-enriched G-extension of $(\mathcal{C}, \mathcal{F}^Z)$ is a triple $(\mathcal{D}, \widetilde{\mathcal{F}}^Z, \alpha)$ such that

- $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ is an ordinary G-extension of $\mathcal{C} = \mathcal{D}_e$;
- $\mathcal{F}^Z : \mathcal{V} \to \mathsf{Rep}(G)' \subset Z(\mathcal{D})$ is a $\mathcal{V}$-enrichment of $\mathcal{D}$ that lands in the Müger centralizer of the canonical copy of $\mathsf{Rep}(G) \subset Z(\mathcal{D})$; and
- α is a natural isomorphism

$$\begin{tikzcd} \mathcal{V} \arrow[rr, dashed, "\widetilde{\mathcal{F}}^Z"] \arrow[d, "\mathcal{F}^Z"'] & & \mathsf{Rep}(G)' \arrow[d, hook] \\ Z(\mathcal{C}) \arrow[rr, phantom, "\xRightarrow{\alpha}"] \arrow[dr, "i"'] & & Z(\mathcal{D}) \arrow[dl, "\mathrm{Forget}_{\mathcal{C}}"] \\ & Z_{\mathcal{C}}(\mathcal{D}), & \end{tikzcd} \tag{4.1}$$

where $\mathrm{Forget}_{\mathcal{C}} : Z(\mathcal{D}) \to Z_{\mathcal{C}}(\mathcal{D})$ denotes the forgetful functor.

Observe that $\mathcal{V}$-enriched G-extensions of $(\mathcal{C}, \mathcal{F}^Z)$ form a 2-groupoid that admits an obvious forgetful 2-functor to the 2-groupoid of ordinary G-extensions of $\mathcal{C}$ as an ordinary fusion category. This forgetful functor is fully faithful at the level of

2-morphisms and a discrete fibration. Hence, by similar arguments to those in Section 3.2.2, the homotopy fiber over a fixed ordinary G-extension $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ of $\mathcal{C}$ is 0-truncated and in bijective correspondence with the strict fiber over $\mathcal{D}$, that is, set of tensor functors $\mathcal{F}^Z : \mathcal{V} \to \mathsf{Rep}(G)' \subset Z(\mathcal{D})$ such that $\mathrm{Forget}_{\mathcal{C}} \circ \widetilde{\mathcal{F}}^Z = i \circ \mathcal{F}^Z$ on the nose.

Remark 4.2. Given an ordinary G-extension $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ of our $\mathcal{V}$-fusion category $(\mathcal{C}, \mathcal{F}^Z)$, choosing a functor $\widetilde{\mathcal{F}}^Z : \mathcal{V} \to \mathsf{Rep}(G)' \subset Z(\mathcal{D})$ in the strict fiber over $\mathcal{D}$ is equivalent to choosing for all $v \in \mathcal{V}$ coherent lifts of the half-braidings for $\mathcal{F}^Z(v)$ with $\mathcal{C}$ to all of $\mathcal{D}$.

We now use the ENO extension theory for fusion categories [19], together with the results from [21], to give several equivalent characterizations of the set of compatible $\mathcal{V}$-enrichments on a fixed ordinary G-extension $\mathcal{D}$ of our $\mathcal{V}$-fusion category $(\mathcal{C}, \mathcal{F}^Z)$.

4.1 Classification in terms of monoidal 2-functors

Definition 4.3. The *$\mathcal{V}$-Brauer–Picard 2-groupoid* $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C}, \mathcal{F}^Z)$ of the $\mathcal{V}$-fusion category $(\mathcal{C}, \mathcal{F}^Z)$ is obtained by taking the ordinary unenriched Brauer–Picard 2-groupoid $\underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ and imposing extra structure.

- Objects in $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C}, \mathcal{F}^Z)$ are invertible $\mathcal{C} - \mathcal{C}$ bimodules $\mathcal{M}$ equipped with natural isomorphisms $\eta_{a,m} : m \triangleleft F_{\mathcal{C}}(a) \to F_{\mathcal{C}}(a) \triangleright m$ satisfying (2.4), (2.5), and (2.6).
- 1-Morphisms are bimodule equivalences $E : \mathcal{M} \to \mathcal{N}$ satisfying (2.8).
- 2-Morphisms are all bimodule natural isomorphisms.

We now endow $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C}, \mathcal{F}^Z)$ with the structure of a categorical 2-group (3-group) by lifting the monoidal structure on $\underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$. Observe that there is an obvious forgetful 2-functor $\mathrm{Forget}_{\mathcal{V}} : \underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C}, \mathcal{F}^Z) \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ that forgets the extra structure, and $\mathrm{Forget}_{\mathcal{V}}$ is fully faithful at the level of 2-morphisms and a discrete fibration.

We now define a monoidal structure on objects in $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C}, \mathcal{F}^Z)$ as the relative Deligne tensor product in $\underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ from Definition 2.14, together with the centering morphism defined from the vertical composition on 2-morphisms in BrdFus from (2.7). The monoidal product of objects in $\underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ is defined by a universal property, so there is not just one composite; there is a contractible choice. We observe that when these bimodules are in the image of the forgetful 2-functor $\mathrm{Forget}_{\mathcal{V}}$, there is a choice of composite $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$, which is also in the image of $\mathrm{Forget}_{\mathcal{V}}$ by construction. Following [22], we get an associator and pentagonator for $\underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ from the universal property defining the relative Deligne product. By the universal property, these associators lift to $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C},\mathcal{F}^{Z})$, and since $\mathrm{Forget}_{\mathcal{V}}$ is fully faithful on 2-morphisms, so does the pentagonator. This also means the forgetful 2-functor $\mathrm{Forget}_{\mathcal{V}}$ is automatically a monoidal 2-functor.

Remark 4.4. We expect that $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C},\mathcal{F}^{Z})$ is monoidally 2-equivalent to the core of the endomorphism monoidal 2-category of the 1-morphism $(\mathcal{C},\mathcal{F}^{Z}) \in \mathsf{BrdFus}(\mathcal{V} \to \mathsf{Vec})$. We leave this verification to the interested reader.

Observe that there is a 2-groupoid of monoidal 2-functors $\mathrm{Hom}(\underline{G} \to \underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C},\mathcal{F}^{Z}))$, and this 2-groupoid admits a 2-functor $U := (\mathrm{Forget}_{\mathcal{V}})_*$ to the 2-groupoid of monoidal 2-functors $\mathrm{Hom}(\underline{G} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C}))$. By Theorem 2.15, this latter 2-groupoid is equivalent to the 2-groupoid $\mathsf{Ext}(\mathcal{C},G)$ of G-extensions of $\mathcal{C}$ as an ordinary fusion category. Now, fixing an ordinary G-extension $\mathcal{D}$ of $\mathcal{C}$, we get a corresponding monoidal 2-functor $\underline{\pi} : \underline{G} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$. Similar to Example 3.18, the homotopy fiber $\mathsf{hoFib}_{\pi}(U)$ is 0-truncated, that is, a set. Moreover, since U is a discrete fibration, this set is in bijection to the strict fiber $\mathsf{stFib}_{\pi}(U)$ whose elements are monoidal 2-functors $\underline{\pi}^{\mathcal{V}} : \underline{G} \to \underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C},\mathcal{F}^{Z})$ such that $\mathrm{Forget}_{\mathcal{V}} \circ \underline{\pi}^{\mathcal{V}} = \underline{\pi}$ *on the nose*.

$$\begin{tikzcd} & \underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C},\mathcal{F}^{Z}) \arrow[d, "\mathrm{Forget}_{\mathcal{V}}"] \\ \underline{G} \arrow[ur, "\underline{\pi}^{\mathcal{V}}"] \arrow[r, "\underline{\pi}"] & \underline{\underline{\mathsf{BrPic}}}(\mathcal{C}) \end{tikzcd}$$

We call this set the *set of lifts* of $\underline{\pi}$ to $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C},\mathcal{F}^{Z})$.

We now prove a version Theorem 2.15 for $\mathcal{V}$-fusion categories.

Theorem 4.5. Let $(\mathcal{C},\mathcal{F}^{Z})$ be a $\mathcal{V}$-fusion category and $\mathcal{D}$ an ordinary G-extension of $\mathcal{C}$. Let $\underline{\pi} : \underline{G} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ be any monoidal 2-functor corresponding to $\mathcal{D}$ under the equivalence of 2-groupoids $\mathsf{Ext}(\mathcal{C},G) \cong \mathrm{Hom}(\underline{G} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C}))$ afforded by Theorem 2.15. The set of $\mathcal{V}$-enrichments $\mathcal{F}^{Z}_{\mathcal{D}} : \mathcal{V} \to Z(\mathcal{D})$ compatible with the enrichment $\mathcal{F}^{Z} : \mathcal{V} \to Z(\mathcal{C})$ is in bijective correspondence with the set of lifts of $\underline{\pi} : \underline{G} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ to $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{V}}(\mathcal{C},\mathcal{F}^{Z})$.

Proof. Suppose we can lift the $\mathcal{V}$-enrichment of $\mathcal{C}$ to $\mathcal{D}$. We define morphisms $\eta_{v,m} : m \triangleleft F_{\mathcal{C}}(v) \to F_{\mathcal{C}}(v) \triangleright m$ for each $m \in \mathcal{D}_g$, where $F_{\mathcal{C}} : \mathcal{V} \to Z(\mathcal{C}) \to \mathcal{C}$ as follows. A lift $\widetilde{\mathcal{F}} : \mathcal{V} \to Z(\mathcal{D})$ applied to a $v \in \mathcal{V}$ can be viewed as $\widetilde{\mathcal{F}}(v) = (F_{\mathcal{C}}(v), \sigma_{\bullet,F_{\mathcal{C}}(v)})$, where $\sigma_{\bullet,F_{\mathcal{C}}(v)}$ is a half-braiding for $F_{\mathcal{C}}(v)$ with $d \in \mathcal{D}$. We define $\eta_{v,d} := \sigma_{d,F_{\mathcal{C}}(v)} : d \otimes F_{\mathcal{C}}(v) \to F_{\mathcal{C}}(v) \otimes d$.

The fact that $\widetilde{\mathcal{F}} : \mathcal{V} \to Z(\mathcal{D})$ is a braided monoidal functor ensures that $\eta_{v,d}$ makes the diagrams (2.4), (2.5), and (2.6) commute. This means we can lift the image of the monoidal 2-functor $\underline{G} \to \underline{\mathsf{BrPic}}(\mathcal{C})$ to $\underline{\mathsf{BrPic}}^{\mathcal{V}}(\mathcal{C}, \mathcal{F}^{Z})$ at the level of 1-morphisms. To lift at the level of 2-morphisms, recall that $\otimes$ induces a bimodule equivalence $\mathcal{D}_g \boxtimes_{\mathcal{C}} \mathcal{D}_h \to \mathcal{D}_{gh}$. We need to show that this bimodule equivalence is a morphism in $\underline{\mathsf{BrPic}}^{\mathcal{V}}(\mathcal{C}, \mathcal{F}^{Z})$. Given objects $d_g \in \mathcal{D}_g, d_h \in \mathcal{D}_h$, we need to check the following diagram commutes:

$$\begin{tikzcd}
\otimes\Big((d_g \boxtimes_{\mathcal{C}} d_h) \triangleleft F_{\mathcal{C}}(v)\Big) = d_g \otimes (d_h \otimes F_{\mathcal{C}}(v)) \arrow[r] \arrow[d] & \otimes\Big(F_{\mathcal{C}}(v) \triangleright (d_g \boxtimes_{\mathcal{C}} d_h)\Big) = (F_{\mathcal{C}}(v) \otimes d_g) \otimes d_h \arrow[d] \\
\otimes(d_g \boxtimes_{\mathcal{C}} d_h) \triangleleft F_{\mathcal{C}}(v) = (d_g \otimes d_h) \otimes F_{\mathcal{C}}(v) \arrow[r] & F_{\mathcal{C}}(v) \triangleright \Big(\otimes(d_g \boxtimes_{\mathcal{C}} d_h)\Big) = F_{\mathcal{C}}(v) \otimes (d_g \otimes d_h),
\end{tikzcd} \tag{4.2}$$

where the top isomorphism is that from (2.7). This now follows immediately from the associativity of a half-braiding.

Conversely, given a $\underline{\pi}^{\mathcal{V}} : \underline{G} \to \underline{\mathsf{BrPic}}^{\mathcal{V}}(\mathcal{C}, \mathcal{F}^{Z})$ such that $\mathrm{Forget}_{\mathcal{V}} \circ \underline{\pi}^{\mathcal{V}} = \underline{\pi}$, we need to extend the half-braiding of $\mathcal{F}^{Z}(v)$ with $\mathcal{C}$ to all of $\mathcal{D}$. We simply use η^{g} on $\mathcal{D}_g$ as our half-braiding:

$$\eta^{g}_{v,m_g} : m_g \triangleleft F_{\mathcal{D}_g}(v) = m_g \otimes \mathcal{F}(v) \to \mathcal{F}(v) \otimes m_g = F_{\mathcal{D}_g}(v) \triangleright m_g.$$

Now, one uses the commutativity of (2.4), (2.5), (2.6) and (4.2) to verify that this is a well-defined half-braiding with all of $\mathcal{D}$.

Finally, one verifies these two constructions are mutually inverse. ■

4.2 Classification in terms of G-equivariant structures on $\mathcal{F}^{Z}$

We now show that given a $\mathcal{V}$-fusion category $(\mathcal{C}, \mathcal{F}^{Z})$ and an ordinary G-extension $\mathcal{D}$ of $\mathcal{C}$, the set of possible compatible $\mathcal{V}$-enrichments on $\mathcal{D}$ is in canonical bijection with G-equivariant structures on the classifying functor $\mathcal{F}^{Z} : \mathcal{V} \to Z(\mathcal{C})$ with respect to the categorical action $\underline{\rho} : \underline{G} \to \underline{\mathrm{Aut}}^{br}_{\otimes}(Z(\mathcal{C}))$ induced from the G-extension $\mathcal{C} \subseteq \mathcal{D} = \bigoplus_g \mathcal{D}_g$. (Recall from Example 3.10 that lifts of $\mathcal{F}^{Z} : \mathcal{V} \to Z(\mathcal{C})$ to $Z(\mathcal{C})^{G}$ naturally form a set.)

Theorem 4.6. The lifts $\widetilde{\mathcal{F}}^{Z} : \mathcal{V} \to Z(\mathcal{D})$ that are compatible with $\mathcal{F}^{Z} : \mathcal{V} \to Z(\mathcal{C})$ are in bijective correspondence with lifts $\widetilde{\mathcal{F}}^{Z} : \mathcal{V} \to Z(\mathcal{C})^{G}$, which satisfy $\mathrm{Forget}_{G} \circ \widetilde{\mathcal{F}}^{Z} = \mathcal{F}^{Z}$, where $\mathrm{Forget}_G : Z(\mathcal{C})^G \to Z(\mathcal{C})$ forgets the G-equivariant structure.

$$\begin{tikzcd} \mathcal{V} \arrow[rr, dashed, "\widetilde{\mathcal{F}}^Z"] \arrow[dr, "\mathcal{F}^Z"'] & & Z(\mathcal{C})^G \arrow[dl, "\mathrm{Forget}_G"] \\ & Z(\mathcal{C}) & \end{tikzcd} \tag{4.3}$$

Proof. We insert the commutative diagram (2.3) based on [21] into (4.1) to get the following diagram:

$$\begin{tikzcd} \mathcal{V} \arrow[rr, dashed, bend left, "\widetilde{\mathcal{F}}^Z"] \arrow[d, "\mathcal{F}^Z"'] & Z(\mathcal{C})^G \arrow[r, leftrightarrow, "\cong"] \arrow[dl, "\mathrm{Forget}_G"'] \arrow[d, hook] & \mathrm{Rep}(G)' \arrow[d, hook] \\ Z(\mathcal{C}) \arrow[dr, hook, "i"'] & Z_{\mathcal{C}}(\mathcal{D})^G \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\mathrm{Forget}_G"] & Z(\mathcal{D}) \arrow[dl, "\mathrm{Forget}_{\mathcal{C}}"] \\ & Z_{\mathcal{C}}(\mathcal{D}) & \end{tikzcd} \tag{4.4}$$

We see that the set of lifts $\widetilde{\mathcal{F}}^Z : \mathcal{V} \to \mathrm{Rep}(G)' \cap Z(\mathcal{D})$ that are compatible with $\mathcal{F}^Z$ are in bijective correspondence with lifts $\widetilde{\mathcal{F}}^Z : \mathcal{V} \to Z(\mathcal{C})^G$, which satisfy $i \circ \mathrm{Forget}_G \circ \widetilde{\mathcal{F}}^Z = i \circ \mathcal{F}^Z$. Since i is faithful on both objects and morphisms, we can cancel it from the left on both sides of the equation and the result follows. ■

Thus, to classify enriched extensions, we must solve the equivariant lifting problem for the data given by the initial enrichment and the extension. In other words, given an (oplax) braided (strongly unital) monoidal functor $\mathcal{F}^Z : \mathcal{V} \to Z(\mathcal{C})$ and a categorical action $\underline{\rho} : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}^{\mathrm{br}}(Z(\mathcal{C}))$, we need to find all the G-equivariant structures on $\mathcal{F}^Z$. We will formalize this notion in Definition 5.3 in the next section.

5 The Equivariant Functor Lifting Problem

In this section, we study the equivariant functor lifting problem, showing lifts are in bijection with splittings of a certain exact sequence. Our approach is similar to [4, Section 3]. We do so in greater generality than needed for (4.3) above, since our results are significantly more general.

For this section, $\mathcal{V}, \mathcal{W}$ will denote linear monoidal categories (which are not necessarily braided!) and $(\mathcal{F}, \varphi, \varepsilon) : \mathcal{V} \to \mathcal{W}$ denotes an oplax monoidal functor (which need not be strongly unital!), where $\varphi = \{\varphi_{u,v} : \mathcal{F}(uv) \to \mathcal{F}(u)\mathcal{F}(v)\}_{u,v \in \mathcal{V}}$ is the oplaxitor and $\varepsilon : \mathcal{F}(1_{\mathcal{V}}) \to 1_{\mathcal{W}}$ is the counit.

Assumption 5.1. Notice that $F(1_{\mathcal{V}}) \in \mathcal{W}$ is a coalgebra object with comultiplication $\Delta := \varphi_{1_{\mathcal{V}},1_{\mathcal{V}}}$ and counit ε. For this section, we assume $\mathcal{F}(1_{\mathcal{V}})$ is connected, that is, $\mathcal{W}(\mathcal{F}(1_{\mathcal{V}}) \to 1_{\mathcal{W}}) = \mathbb{C}\varepsilon$.

We further suppose $(\underline{\rho}, \mu) : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathcal{W})$ is a categorical action of the finite group G. We write $g = \rho_g$ for notational simplicity, and we write ψ^g for its tensorator. Our convention for the tensorator μ for $\underline{\rho}$ is $\mu_{g,h} : g \circ h \Rightarrow gh$.

5.1 The first obstruction

Definition 5.2. We consider the following categorical groups.

- $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{W})$ is the categorical group of (strong) monoidal auto-equivalences of $\mathcal{W}$. Thought of as a monoidal category, objects are monoidal auto-equivalences of $\mathcal{W}$, and morphisms are monoidal natural isomorphisms.
- $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{W}|\mathcal{F})$ is the categorical group defined as follows: objects are triples $(\alpha, \psi^{\alpha}, \lambda^{\alpha})$, where $(\alpha, \psi^{\alpha}) \in \underline{\mathrm{Aut}}_{\otimes}(\mathcal{W})$ is a monoidal auto-equivalence of $\mathcal{W}$ (here, ψ^{α} is the tensorator of α), and $\lambda^{\alpha} : \mathcal{F} \Rightarrow \alpha \circ \mathcal{F}$ is an (oplax) monoidal natural isomorphism. The 1-composition is strict and defined as

$$(\alpha, \psi^{\alpha}, \lambda^{\alpha}) \circ (\beta, \psi^{\beta}, \lambda^{\beta}) := (\alpha \circ \beta, \psi^{\alpha} \circ \alpha(\psi^{\beta}), \lambda^{\alpha} \circ \alpha(\lambda^{\beta})).$$

  The 2-morphisms $\eta : (\alpha, \psi^{\alpha}, \lambda^{\alpha}) \Rightarrow (\beta, \psi^{\beta}, \lambda^{\beta})$ are all monoidal natural isomorphisms $\eta : (\alpha, \psi^{\alpha}) \Rightarrow (\beta, \psi^{\beta})$ such that $(\eta \circ \mathrm{id}_{\mathcal{F}}) \circ \lambda^{\alpha} = \lambda^{\beta}$.
- $\underline{\mathrm{Stab}}_{\otimes}(\mathcal{F})$ is the full categorical subgroup of $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{W})$ generated by the image of $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{W}|\mathcal{F})$ under the forgetful functor $(\alpha, \psi^{\alpha}, \lambda^{\alpha}) \mapsto (\alpha, \psi^{\alpha})$.

Definition 5.3. Let $\rho : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathcal{W})$, $g \mapsto \rho_g$ be a categorical action, and $\mathcal{F} : \mathcal{V} \to \mathcal{W}$ an oplax monoidal functor. A *G-equivariant structure* on $\mathcal{F}$ is a lifting

$$\begin{tikzcd} & \underline{\mathrm{Aut}}_{\otimes}(\mathcal{W}|\mathcal{F}) \arrow[d, "\mathrm{Forget}_{\mathcal{F}}"] \\ \underline{G} \arrow[ur, dashed, "\underline{\widetilde{\rho}}"] \arrow[r, "\underline{\rho}"'] & \underline{\mathrm{Aut}}_{\otimes}(\mathcal{W}), \end{tikzcd} \tag{5.1}$$

which satisfies $\mathrm{Forget}_{\mathcal{F}} \circ \underline{\widetilde{\rho}} = \underline{\rho}$ on the nose.

Hence, in order to find a lifting $\underline{\widetilde{\rho}} : \underline{G} \to \underline{\mathrm{Aut}}(\mathcal{W}|\mathcal{F})$, it is necessary that for each $g \in G$, there exists a monoidal natural isomorphism $\lambda^g : \mathcal{F} \Rightarrow g \circ \mathcal{F}$. We call the existence of such a λ^g for each $g \in G$ the *first obstruction* to the equivariant functor lifting problem. We say *the first obstruction vanishes* if such a λ^g exists for each $g \in G$.

5.2 The second obstruction

We now assume that the first obstruction to the equivariant lifting problem vanishes, that is, for every $g \in G$, there exists a monoidal natural isomorphism $\lambda^g : \mathcal{F} \Rightarrow g \circ \mathcal{F}$. We now give a necessary and sufficient condition for the isomorphisms $(\lambda^g)_{g\in G}$ to assemble to a lift $\underline{\widetilde{\rho}} : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathcal{W}|\mathcal{F})$. We call this condition the *second obstruction* to the equivariant functor lifting problem.

Recall that the adjoint to the forgetful functor $\mathrm{Forget}_G : \mathcal{W}^G \to \mathcal{W}$ is $I : \mathcal{W} \to \mathcal{W}^G$ by $w \mapsto \bigoplus g(w)$ and $f \in \mathcal{W}(w_1 \to w_2)$ maps to $I(f)_{g,h} := \delta_{g,h} \cdot g(f)$. Observe that given $w \in \mathcal{W}$, $f : I(w) \to I(w)$ is G-equivariant if and only if the following diagram commutes for all $g, h, k \in G$:

$$\begin{tikzcd} g(k(w)) \arrow[r, "\mu^w_{g,k}"] \arrow[d, "g(f_{h,k})"] & (gk)(w) \arrow[d, "f_{gh,gk}"] \\ g(h(w)) \arrow[r, "\mu^w_{g,h}"] & (gh)(w) \end{tikzcd} \qquad \forall g, h, k \in G, \tag{5.2}$$

where $f_{h,k} : k(w) \to h(w)$ is the (h,k)-component map of f. The functor I is endowed with an oplax monoidal structure $\nu^I_{w_1,w_2} \in \mathcal{W}^G(I(w_1 \otimes w_2) \to I(w_1) \otimes I(w_2))$ given componentwise by

$$\bigoplus_{g\in G} \psi^g_{w_1,w_2} : \bigoplus_{g\in G} g(w_1 \otimes w_2) \xrightarrow{\psi^g_{w_1,w_2}} \bigoplus_{g\in G} g(w_1) \otimes g(w_2) \subseteq \bigoplus_{g,h\in G} g(w_1) \otimes h(w_2) \cong I(w_1) \otimes I(w_2).$$

Remark 5.4. In addition to $\mathcal{F}(1_{\mathcal{V}})$ being a coalgebra with comultiplication Δ (see Assumption 5.1), notice that $(I \circ \mathcal{F})(1_{\mathcal{V}}) \in \mathcal{W}^G$ is also a coalgebra object with comultiplication given on components by

$$\Lambda^{g,h}_k := \delta_{g=h}\delta_{g=k} \cdot \psi^g_{\mathcal{F}(1_{\mathcal{V}}),\mathcal{F}(1_{\mathcal{V}})} \circ g(\Delta) : k(\mathcal{F}(1_{\mathcal{V}})) \to g(\mathcal{F}(1_{\mathcal{V}})) \otimes h(\mathcal{F}(1_{\mathcal{V}}))$$

and counit given on components by $\varepsilon_g := g(\varepsilon^{\mathcal{F}}) : g(\mathcal{F}(1_{\mathcal{V}})) \to 1_{\mathcal{W}}$.

We define $\iota : \mathrm{Aut}_{\otimes}(\mathcal{F}) \to \mathrm{Aut}_{\otimes}(I \circ \mathcal{F})$ by $\iota(f)^v := I(f^v) \in \mathcal{W}^G(I(\mathcal{F}(v)) \to I(\mathcal{F}(v)))$. To verify that $\iota(f)$ is oplax monoidal, we see the outside square of the following diagram commutes, as the inner squares both commute:

$$\begin{array}{ccccc}
I(\mathcal{F}(v_1 \otimes v_2)) & \xrightarrow{I(\varphi^{v_1,v_2})} & I(\mathcal{F}(v_1) \otimes \mathcal{F}(v_2)) & \xrightarrow{\nu^{F(v_1),F(v_2)}} & I(\mathcal{F}(v_1)) \otimes I(\mathcal{F}(v_2)) \\
\downarrow{\scriptstyle I(f^{v_1 \otimes v_2})} & & \downarrow{\scriptstyle I(f^{v_1} \otimes f^{v_2})} & & \downarrow{\scriptstyle I(f^{v_1}) \otimes I(f^{v_2})} \\
I(\mathcal{F}(v_1 \otimes v_2)) & \xrightarrow{I(\varphi^{v_1,v_2})} & I(\mathcal{F}(v_1) \otimes \mathcal{F}(v_2)) & \xrightarrow{\nu^{\mathcal{F}(v_1),\mathcal{F}(v_2)}} & I(\mathcal{F}(v_1)) \otimes I(\mathcal{F}(v_2)).
\end{array}$$

The following lemma is similar to [4, Lemma 3.2]. We provide a proof for completeness and convenience of the reader.

Lemma 5.5. Suppose $\eta \in \mathrm{Aut}_{\otimes}(I \circ \mathcal{F})$.

1. For $h, k \in G$, $\eta^v_{h,k} : k(\mathcal{F}(v)) \to h(\mathcal{F}(v))$ is equal to $\eta^v_{h,k} = \mu^{\mathcal{F}(v)}_{k,k^{-1}h} \circ k(\eta^v_{k^{-1}h,e}) \circ (\mu^{\mathcal{F}(v)}_{k,e})^{-1}$. Hence, η^v is completely determined by its components $\eta^v_{g,e} : g(\mathcal{F}(v)) \to \mathcal{F}(v)$ for $v \in \mathcal{V}$.
2. There is a unique $g \in G$ such that $\eta^{1_{\mathcal{V}}}_{g,e} \neq 0$, and $\eta^{1_{\mathcal{V}}}_{g,e} : \mathcal{F}(1_{\mathcal{V}}) \to g(\mathcal{F}(1_{\mathcal{V}}))$ is a coalgebra isomorphism.
3. For every $h \in G$, there are unique $g, k \in G$ such that $\eta^v_{g,h} \neq 0 \neq \eta^v_{h,k}$ for all $v \in \mathcal{V}$. These g, k are independent of $v \in \mathcal{V}$.

Proof. To prove (1), since $\eta^v : I(\mathcal{F}(v)) \Rightarrow I(\mathcal{F}(v))$ is G-equivariant, replacing h, k by $g^{-1}h, g^{-1}k$, respectively, in (5.2) for $f = \eta^v$ gives

$$\eta^v_{h,k} \circ \mu^{\mathcal{F}(v)}_{g,g^{-1}k} = \mu^{\mathcal{F}(v)}_{g,g^{-1}h} \circ g(\eta^v_{g^{-1}h,g^{-1}k}) \qquad \forall g, h, k \in G.$$

Now, setting $g = k$ gives the desired formula.

To prove (2), we first note that for each $g \in G$, there is a scalar $\gamma_g \in \mathbb{C}$ such that $g(\varepsilon) \circ \eta^{1_{\mathcal{V}}}_{g,e} = \gamma_g \cdot \varepsilon \in \mathcal{C}(\mathcal{F}(1_{\mathcal{V}}) \to 1_{\mathcal{W}}) = \mathbb{C} \cdot \varepsilon$. Looking at the e-component of the counitality axiom

$$\varepsilon^I \circ I(\varepsilon^{\mathcal{F}}) = \varepsilon^I \circ I(\varepsilon^{\mathcal{F}}) \circ \sigma^{1_{\mathcal{V}}} \in \mathcal{W}^G(I(\mathcal{F}(1_{\mathcal{V}})) \to 1_{\mathcal{W}})$$

gives us the identity

$$\varepsilon = \sum_{h \in G} h(\varepsilon^{\mathcal{F}}) \circ \eta^{1_{\mathcal{V}}}_{h,e} = \left(\sum_{h \in G} \gamma_h\right) \varepsilon,$$

which implies $\sum_h \gamma_h = 1$. Fix $h \in G$ such that $\gamma_h \neq 0$. For $g \neq h$, looking at the component $\Lambda_e^{h,g} : \mathcal{F}(1_\mathcal{V}) \to h(\mathcal{F}(1_\mathcal{V})) \otimes g(\mathcal{F}(1_\mathcal{V}))$ yields the identity

$$(\eta_{h,e}^{1_\mathcal{V}} \otimes \eta_{g,e}^{1_\mathcal{V}}) \circ \psi_e^{\mathcal{F}(1_\mathcal{V}),\mathcal{F}(1_\mathcal{V})} \circ \Delta = \delta_{h=g}\psi_g^{\mathcal{F}(1_\mathcal{V}),\mathcal{F}(1_\mathcal{V})} \circ g(\Delta) \circ \eta_{g,e}^{1_\mathcal{V}} = 0.$$

Postcomposing with $h(\varepsilon) \otimes \mathrm{id}_{g(\mathcal{F}(1_\mathcal{V}))}$ yields

$$0 = ((\eta_{h,e}^{1_\mathcal{V}} \circ h(\varepsilon^\mathcal{F})) \otimes \eta_{g,e}^{1_\mathcal{V}}) \circ \Delta = \gamma_h \cdot (\varepsilon^\mathcal{F} \otimes \eta_{g,e}^{1_\mathcal{V}}) \circ \Delta = \gamma_h \cdot \eta_{g,e}^{1_\mathcal{V}}.$$

Since $\gamma_h \neq 0$, we conclude $\eta_{g,e}^{1_\mathcal{V}} = 0$ whenever $g \neq h$, proving (2). Notice that this also proves $\gamma_h = 1$. That $\eta_{g,e}^{1_\mathcal{V}} : \mathcal{F}(1_\mathcal{V}) \to g(\mathcal{F}(1_\mathcal{V}))$ is a coalgebra isomorphism follows immediately by looking at components as above.

Now, (3) follows by (1) and (2) using monoidality of η. Indeed, for $v \in \mathcal{V}$, we have $v = 1_\mathcal{V} \otimes v$ (suppressing unitors), so the components of $\eta^v \in \mathrm{End}_{\mathcal{W}^G}((I \circ \mathcal{F})(v) = \bigoplus_g g(\mathcal{F}(v)))$ satisfy the following commuting diagram below:

$$\begin{array}{ccc}
h(\mathcal{F}(v)) & \xrightarrow{\psi^h_{\mathcal{F}(1_\mathcal{V}),\mathcal{F}(v)} \circ h(\varphi_{1_\mathcal{V},v})} & h(\mathcal{F}(1_\mathcal{V})) \otimes h(\mathcal{F}(v)) \\
\big\downarrow \eta^v_{g,h} & & \big\downarrow \eta^{1_\mathcal{V}}_{g,h} \otimes \eta^v_{g,h} \\
g(\mathcal{F}(v)) & \xrightarrow{\psi^g_{\mathcal{F}(1_\mathcal{V}),\mathcal{F}(v)} \circ g(\varphi_{1_\mathcal{V},v})} & g(\mathcal{F}(1_\mathcal{V})) \otimes g(\mathcal{F}(v)).
\end{array}$$

Notice that the map $\psi^g_{\mathcal{F}(1_\mathcal{V}),\mathcal{F}(v)} \circ g(\varphi_{1_\mathcal{V},v})$ has a left inverse for every $v \in \mathcal{V}$, namely $(g(\varepsilon) \otimes \mathrm{id}_{\mathcal{F}(v)}) \circ (\psi^g_{\mathcal{F}(1_\mathcal{V}),\mathcal{F}(v)})^{-1}$. This implies that $\eta^v_{g,h} = 0$ whenever $\eta^{1_\mathcal{V}}_{g,h} = 0$. ■

Lemma 5.6. The function $\pi : \mathrm{Aut}_\otimes(I \circ \mathcal{F}) \to G$ given by setting $\pi(\eta)$ to be the unique g such that $\eta^{1_\mathcal{V}}_{g^{-1},e} \neq 0$ gives a well-defined group homomorphism.

Proof. Suppose $\eta, \xi \in \mathrm{Aut}^\otimes_{\mathcal{W}^G}(I \circ F)$, and consider $\eta \circ \xi$. Then, $\pi(\eta \circ \xi)$ is the unique element $g \in G$ such that $(\eta \circ \xi)^{1_\mathcal{V}}_{g^{-1},e} \neq 0$. We calculate that

$$(\eta \circ \xi)^{1_\mathcal{V}}_{g^{-1},e} = \sum_{h \in G} \eta^{1_\mathcal{V}}_{g^{-1},h} \circ \xi^{1_\mathcal{V}}_{h,e} = \eta^{1_\mathcal{V}}_{g,\pi(\xi)^{-1}} \circ \xi^{1_\mathcal{V}}_{\pi(\xi)^{-1},e}.$$

By (5.2), we see that $\eta^{1_\mathcal{V}}_{g^{-1},\pi(\xi)^{-1}} \neq 0$ if and only if $\eta^{1_\mathcal{V}}_{\pi(\xi)g^{-1},e} \neq 0$. Hence, $(\pi(\xi)g^{-1})^{-1} = \pi(\eta)$, which immediately implies $\pi(\eta \circ \xi) = g = \pi(\eta) \cdot \pi(\xi)$. ■

Lemma 5.7. For every $\eta \in \pi^{-1}(g^{-1})$, $\theta_v := \eta^v_{g,e} : \mathcal{F}(v) \to g(\mathcal{F}(v))$ gives an monoidal natural isomorphism $\theta : \mathcal{F} \Rightarrow g \circ \mathcal{F}$. Moreover, every monoidal natural isomorphism

$\mathcal{F} \Rightarrow g \circ \mathcal{F}$ arises in this way. Hence, $\pi^{-1}(g^{-1})$ is in bijective correspondence with monoidal natural isomorphisms $\theta : \mathcal{F} \Rightarrow g \circ \mathcal{F}$.

Proof. First, if $\eta \in \pi^{-1}(g^{-1})$, then the following diagram commutes for all $g \in G$ as η is an oplax monoidal automorphism of $I \circ \mathcal{F}$:

$$\begin{tikzcd}
\mathcal{F}(uv) \arrow[r, "\varphi_{u,v}"] \arrow[d, "\eta^{uv}_{g,e}"] & \mathcal{F}(u) \otimes \mathcal{F}(v) \arrow[d, "\eta^{u}_{g,e} \otimes \eta^{v}_{g,e}"] \\
g(\mathcal{F}(uv)) \arrow[r, "\psi^{g}_{\mathcal{F}(u),\mathcal{F}(v)} \circ g(\varphi_{u,v})"] & g(\mathcal{F}(u)) \otimes g(\mathcal{F}(v)).
\end{tikzcd}$$

Notice this is exactly the condition that $\theta : \mathcal{F} \Rightarrow g \circ \mathcal{F}$ is oplax monoidal. Conversely, if $\theta : \mathcal{F} \Rightarrow g \circ \mathcal{F}$ is an monoidal natural isomorphism, then defining

$$\eta^{v}_{h,k} := \delta_{g=k^{-1}h} \cdot \mu^{\mathcal{F}(v)}_{k,g} \circ k(\theta_v) \circ (\mu^{\mathcal{F}(v)}_{k,e})^{-1}$$

gives a well-defined $\eta \in \pi^{-1}(g^{-1})$ such that $\eta^{v}_{g,e} = \theta_v$ by construction. ■

Proposition 5.8. The following sequence is exact:

$$1 \longrightarrow \mathrm{Aut}_{\otimes}(\mathcal{F}) \xrightarrow{\iota} \mathrm{Aut}_{\otimes}(I \circ \mathcal{F}) \xrightarrow{\pi} G \longrightarrow 1. \tag{5.3}$$

Proof. The map ι is injective by definition. The map π is surjective by Lemma 5.7. To see $\mathrm{im}(\iota) = \ker(\pi)$, if $\eta \in \ker(\pi)$, then each η^{v} is determined by $\theta^{v} := \eta^{v}_{e,e} : \mathcal{F}(v) \to \mathcal{F}(v)$ by Lemma 5.5, and $\theta : \mathcal{F} \Rightarrow \mathcal{F}$ is a monoidal natural isomorphism such that $\iota(\theta) = \eta$. ■

Theorem 5.9. The set of G-equivariant structures on $\mathcal{F}$ as in (5.1) is in bijective correspondence with splittings of the exact sequence (5.3).

Proof. Suppose $\underline{\widetilde{\rho}}$ is a lift of $\underline{\rho}$, and denote $\underline{\widetilde{\rho}}(g) = (g, \lambda_g)$, where $\lambda^g : \mathcal{F} \Rightarrow g \circ \mathcal{F}$ is a monoidal natural isomorphism. We get a splitting $\sigma : G \to \mathrm{Aut}_{\otimes}(I \circ \mathcal{F})$ by mapping g^{-1} to the element corresponding to λ^g. Conversely, given a splitting σ, $\sigma(g^{-1}) \in \pi^{-1}(g^{-1})$ gives an monoidal natural isomorphism $\lambda^g := \sigma(g^{-1})_{g,e} : \mathcal{F} \Rightarrow g \circ \mathcal{F}$. One now verifies that $\underline{\widetilde{\rho}}(g) := (g, \lambda_g)$ is the desired lift. These two constructions are clearly mutually inverse. ■

5.3 The braided case

We now assume $\mathcal{V}, \mathcal{W}$ are braided monoidal categories and $\mathcal{F} : \mathcal{V} \to \mathcal{W}$ is an oplax braided monoidal functor. We again use Assumption 5.1 that $\mathcal{F}(1_{\mathcal{V}})$ is a connected coalgebra in $\mathcal{W}$.

Definition 5.10. We consider the following categorical groups.

- $\underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathcal{W})$ is the full categorical subgroup of $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{W})$ whose objects are braided (strong) monoidal auto-equivalences of $\mathcal{W}$. Observe that if $(\alpha, \psi^\alpha) \in \underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathcal{W})$, $(\gamma, \psi^\gamma) \in \underline{\mathrm{Aut}}_{\otimes}(\mathcal{W})$, and $\eta : (\alpha, \psi^\alpha) \Rightarrow (\gamma, \psi^\gamma)$ is a monoidal natural isomorphism, then $(\gamma, \psi^\gamma) \in \underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathcal{W})$, as the back face of the following diagram commutes:

$$\begin{tikzcd}
& \gamma(u \otimes v) \arrow[rr, "\psi^\gamma_{u,v}"] \arrow[dd, "\gamma(\beta^{\mathcal{V}}_{u,v})"' near start] & & \gamma(u) \otimes \gamma(v) \arrow[dd, "\beta^{\mathcal{W}}_{\gamma(u),\gamma(v)}"] \\
\alpha(u \otimes v) \arrow[ur, "\eta_{uv}"] \arrow[rr, "\psi^\alpha_{u,v}" near end, crossing over] \arrow[dd, "\alpha(\beta^{\mathcal{V}}_{u,v})"'] & & \alpha(u) \otimes \alpha(v) \arrow[ur, "\eta_u \otimes \eta_v"'] & \\
& \gamma(v \otimes u) \arrow[rr, "\psi^\gamma_{v,u}" near end] & & \gamma(v) \otimes \gamma(u) \\
\alpha(v \otimes u) \arrow[ur, "\eta_{vu}"] \arrow[rr, "\psi^\alpha_{v,u}"] & & \alpha(v) \otimes \alpha(u) \arrow[ur, "\eta_v \otimes \eta_u"'] \arrow[from=uu, "\beta^{\mathcal{W}}_{\alpha(u),\alpha(v)}"' near end, crossing over] &
\end{tikzcd}.$$

  Indeed, the left face commutes since η is natural, the right face commutes since $\beta^{\mathcal{W}}$ is natural, the top and bottom faces commute since η is monoidal, and the front face commutes since α is braided. We conclude the back face must also commute.
- $\underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathcal{W}|\mathcal{F})$ is a the full categorical subgroup of $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{W}|\mathcal{F})$ whose objects are triples $(\alpha, \psi^\alpha, \lambda^\alpha)$, where $(\alpha, \psi^\alpha) \in \underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathcal{W})$.
- $\underline{\mathrm{Stab}}^{\mathrm{br}}_{\otimes}(\mathcal{F})$ is the full categorical subgroup of $\underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathcal{W})$ generated by the image of $\underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathcal{W}|\mathcal{F})$ under the forgetful functor $(\alpha, \psi^\alpha, \lambda^\alpha) \mapsto (\alpha, \psi^\alpha)$.

In this setting, we make the following definition.

Definition 5.11. Let $\rho : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}^{\mathrm{br}}(\mathcal{W})$ be a categorical action and $\mathcal{F} : \mathcal{V} \to \mathcal{W}$ an oplax braided monodal functor. A *G-equivariant structure* on $\mathcal{F}$ is a lifting

$$\begin{tikzcd} & \underline{\mathrm{Aut}}_{\otimes}^{\mathrm{br}}(\mathcal{W}|\mathcal{F}) \arrow[d, "\mathrm{Forget}_{\mathcal{F}}"] \\ \underline{G} \arrow[ur, dashed, "\widetilde{\underline{\rho}}"] \arrow[r, "\underline{\rho}"'] & \underline{\mathrm{Aut}}_{\otimes}^{\mathrm{br}}(\mathcal{W}), \end{tikzcd} \tag{5.4}$$

which satisfies $\mathrm{Forget}_{\mathcal{F}} \circ \widetilde{\underline{\rho}} = \underline{\rho}$ on the nose.

Since $\pi_2(\underline{\mathrm{Aut}}_{\otimes}^{\mathrm{br}}(\mathcal{W})) = \pi_2(\underline{\mathrm{Aut}}_{\otimes}(\mathcal{W}))$ and $\pi_2(\underline{\mathrm{Aut}}_{\otimes}^{\mathrm{br}}(\mathcal{W}|F)) = \pi_2(\underline{\mathrm{Aut}}_{\otimes}(\mathcal{W}|F))$, G-equivariant lifts as in (5.4) are again in bijective correspondence with splittings of the exact sequence (5.3).

6 Examples

In this section, we work out examples of our main Theorems 4.6 and 5.9 above in the $\mathcal{V}$-fusion setting.

6.1 Fully faithful enrichment

Suppose $(\mathcal{C}, \mathcal{F}^{z})$ is a $\mathcal{V}$-fusion category such that $\mathcal{F}^{z}$ is fully faithful. This type of example is particularly important, since every enrichment can be "pushed forward" to a fully faithful enrichment by considering the enrichment over the full subcategory generated by the image of $\mathcal{V}$ in $Z(\mathcal{C})$. We will see that in the fully faithful setting, the G-action on the normal subgroup $\mathrm{Aut}_{\otimes}(\mathcal{F}^{z})$ is trivial, and thus splitting of the short exact sequence (5.3) becomes a 2-cocycle obstruction.

Now, suppose $\mathcal{D}$ is any G-graded extension of $\mathcal{C}$ as an ordinary fusion category, so we get a categorical action $\underline{\rho} : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}^{\mathrm{br}}(Z(\mathcal{C}))$. Assume that $\underline{\rho}$ passes the first obstruction, so that for each $g \in G$, there exists a monoidal natural isomorphism $\lambda^{g} : \mathcal{F} \Rightarrow g \circ \mathcal{F}$. By a direct computation, we see that

$$\omega(g, h) := (\lambda^{gh})^{-1} \circ \mu^{\mathcal{F}}_{g,h} \circ g(\lambda^{h}) \circ \lambda^{g} : \mathcal{F} \Rightarrow \mathcal{F} \tag{6.1}$$

is an element of $\mathrm{Aut}_{\otimes}(\mathcal{F}) \cong \mathrm{Aut}_{\otimes}(\mathrm{id}_{\mathcal{V}})$, which is in turn isomorphic to the group $\widehat{\mathcal{U}(\mathcal{V})}$ of characters on the universal grading group of $\mathcal{V}$. In fact, $\omega \in Z^{2}(G, \widehat{\mathcal{U}(\mathcal{V})})$. Any other choice of λ^{g} for $g \in G$ will give a cohomologous 2-cocycle. We see directly that the second obstruction vanishes if and only if $[\omega] = 0$ in $H^2(G, \widehat{\mathcal{U}(\mathcal{V})})$. Hence, the exact sequence (5.3) is exactly

$$1 \longrightarrow \widehat{\mathcal{U}(\mathcal{V})} \xrightarrow{\iota} \widehat{\mathcal{U}(\mathcal{V})} \times_\omega G \xrightarrow{\pi} G \longrightarrow 1,$$

which splits if and only if $[\omega] = 0$.

Observe that when ρ passes the first obstruction, the 2-cocycle ω in (6.1) automatically vanishes if $\widehat{\mathcal{U}(\mathcal{V})}$ is trivial, in which case there is a unique splitting.

Corollary 6.1. Suppose $(\mathcal{C}, \mathcal{F}^Z)$ is a $\mathcal{V}$-fusion category with $\mathcal{F}^Z$ fully faithful. Let $\mathcal{D}$ be an arbitrary G-graded extension of $\mathcal{C}$ for which the first obstruction vanishes. If $\widehat{\mathcal{U}(\mathcal{V})}$ is trivial, then the $\mathcal{V}$-enrichment has a unique lifting to $\mathcal{D}$.

Example 6.2. If $(\mathcal{C}, \mathcal{F}^Z)$ is a Fib-fusion category and $\mathcal{D}$ is a G-graded extension of $\mathcal{C}$ for which the first obstruction vanishes, then there is a unique lift of the Fib enrichment to $\mathcal{D}$. For an explicit example, one may consider $\mathcal{C} = \mathrm{Ad}(E_8)$ and $\mathcal{D} = E_8$.

6.2 Zesting a trivial extension

For convenience, we assume that $H^4(G, \mathbb{C}^\times) = (1)$. Recall a braided categorical action of G on $Z(\mathcal{C})$ is called *G-stable* if each $g \in G$ acts by the identity functor. Such actions are given by twisting the trivial action by a 2-cocycle $\omega \in H^2(G, \mathrm{Aut}_\otimes(\mathrm{id}_{Z(\mathcal{C})}) = H^2(G, \mathrm{Inv}(Z(\mathcal{C})))$ [19]. Since $H^4(G, \mathbb{C}^\times) = (1)$, we get a G-graded extension $\mathcal{D}$ of $\mathcal{C}$ called $\mathcal{C} \boxtimes_\omega \mathsf{Vec}(G)$, which is $\mathcal{C} \boxtimes \mathsf{Vec}(G)$ as a linear category with the tensor product functor twisted by ω. Twisting the monoidal product by a 2-cocycle in this manner is sometimes called *zesting*; c.f. [6].

For such extensions, for *any* enrichment $(\mathcal{C}, \mathcal{F}^Z)$, the first obstruction always vanishes, namely $g \circ \mathcal{F}^Z \cong \mathcal{F}^Z$ since $g \cong \mathrm{id}_{Z(\mathcal{C})}$. If, in addition, $\mathcal{F}^Z$ is fully faithful (or more generally sends simple objects to simple objects), then we get a restriction map $R : \mathrm{Aut}_\otimes(\mathrm{id}_{Z(\mathcal{C})}) \cong \mathrm{Inv}(Z(\mathcal{C})) \to \mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{V}) \cong \widehat{\mathcal{U}(\mathcal{V})}$ and the 2-cocycle (6.1) corresponds to the pushforward of $R_*\omega \in H^2(G, \widehat{\mathcal{U}(\mathcal{V})})$. Thus, we can extend the enrichment $(\mathcal{C}, \mathcal{F}^Z)$ if and only if $R_*\omega$ is trivial.

For a slightly more explicit example, when $\mathcal{V} = \mathsf{Rep}(N)$ and $\mathcal{C} = \mathsf{Vec}(N)$, we have $\mathrm{Inv}(Z(\mathcal{C})) \cong \widehat{N} \times Z(N)$ and $\widehat{\mathcal{U}(\mathcal{V})} \cong Z(N)$. Then, the pushforward map $R : \widehat{N} \times Z(N) \to Z(N)$ is the canonical projection to the factor $Z(N)$. In particular, for any group with $Z(N) = (1)$ and for any $\omega \in H^2(G, \widehat{N})$ (with the trivial action of G on $\widehat{N}$), we can lift the $\mathsf{Rep}(N)$ enrichment on $\mathsf{Vec}(N)$ to the zested extension $\mathsf{Vec}(N) \boxtimes_\omega \mathsf{Vec}(G)$. In this case, the latter category is actually equivalent to $\mathsf{Vec}(N \times G)$. Since the 3-cocyle obtained from ω via a connecting map in the Lyndon–Hochschild–Serre spectral sequence associated to the short exact sequence $1 \to N \to N \times G \to G \to 1$ is trivial (see [19,]). Indeed, all differentials in the LHS spectral sequence are zero; we thank the referee for pointing this out.

6.3 Fibered enrichments and group theoretical extensions

In this example, we focus on $\mathsf{Rep}(N)$-fibered enrichments (recall Definition 3.21) with $\mathcal{C} = \mathsf{Vec}(N)$ and $\mathcal{D} = \mathsf{Vec}(E)$ for some normal subgroup $N \leq E$ corresponding to a fixed exact sequence

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1. \tag{6.2}$$

We now analyze when we can extend the $\mathsf{Rep}(N)$-fibered enrichment on $\mathsf{Vec}(N)$ to $\mathsf{Vec}(E)$. The first step will be to analyze the categorical action of G on the center, and in particular, how it restricts to the fibered enrichment.

First, from the extension above we directly define a braided categorical action on $\mathsf{Rep}(N)$. Pick a set theoretical section $\lambda : G \to E$ of the quotient map $E \to G$ that we will denote $g \mapsto \lambda_g \in E$. Then, we have $\lambda_g\lambda_h = \lambda_{gh}n_{g,h}$ for some $n_{g,h} \in N$. For each $g \in G$, we define $\alpha_g \in \underline{\mathrm{Aut}}^{br}_{\otimes}(\mathsf{Rep}(N))$ by $\alpha_g(\pi, V) := (\pi(\lambda_g^{-1} \cdot \lambda_g), V)$ on objects, and we set α_g to be the identity on morphisms. This has the obvious structure of a (braided) monoidal functor. We now define monoidal natural isomorphisms $\mu_{g,h} : \alpha_g \circ \alpha_h \to \alpha_{gh}$. For each $(\pi, V) \in \mathsf{Rep}(N)$, consider the linear map $\pi(n_{g,h})$ on the vector space V. Then, we have

$$\pi(n_{g,h})\pi(\lambda_h^{-1}\lambda_g^{-1} - \lambda_g\lambda_h) = \pi(n_{g,h})\pi(n_{g,h}^{-1}\lambda_{gh}^{-1} - \lambda_{gh}n_{g,h}) = \pi(\lambda_{gh}^{-1} - \lambda_{gh})\pi(n_{g,h}). \qquad \forall g, h \in G.$$

Setting $\mu_{g,h} := \{\mu_{g,h}^{(\pi,V)} := \pi(n_{g,h})\}_{(\pi,V)\in\mathsf{Rep}(N)}$, we see $\mu_{g,h} : \alpha_g \circ \alpha_h \to \alpha_{gh}$ gives a monoidal natural isomorphism of functors.

Lemma 6.3. The assignment $g \mapsto \alpha_g \in \underline{\mathrm{Aut}}^{br}_{\otimes}(\mathsf{Rep}(G))$ together with the monoidal natural isomorphisms $\mu_{g,h} : \alpha_g \circ \alpha_h \to \alpha_{gh}$ described above assembles into a categorical action $\alpha : \underline{G} \to \underline{\mathrm{Aut}}^{br}_{\otimes}(\mathsf{Rep}(N))$.

Proof. A quick computation shows that the equation we need to verify for all $g, h, k \in G$ and all representations (π, V) is the cocycle-type equation

$$\pi(n_{gh,k}\lambda_k^{-1}n_{g,h}\lambda_k) = \pi(n_{g,hk}n_{h,k}). \tag{6.3}$$

From the definition of $n_{g,h}$, we have

$$\lambda_g\lambda_h\lambda_k = \lambda_{gh}n_{g,h}\lambda_k = \lambda_{gh}\lambda_k\lambda_k^{-1}n_{g,h}\lambda_k = \lambda_{ghk}n_{gh,k}\lambda_k^{-1}n_{g,h}\lambda_k.$$

On the other hand, we also have

$$\lambda_g\lambda_h\lambda_k = \lambda_g\lambda_{hk}n_{h,k} = \lambda_{ghk}n_{g,hk}n_{h,k}.$$

Comparing these two expressions, we see $n_{gh,k}\lambda_k^{-1}n_{g,h}\lambda_k = n_{g,hk}n_{h,k}$ in N, so (6.3) holds for any representation of N. ■

Now, we consider $\mathsf{Vec}(E)$ as a G-extensions of $\mathsf{Vec}(N)$. This yields a braided categorical action, which we denote $\tilde{\alpha} : \underline{G} \to \underline{\mathrm{Aut}}_\otimes^{br}(Z(\mathcal{C}))$.

Lemma 6.4. The categorical action $\tilde{\alpha}$ restricts on the canonical copy of $\mathsf{Rep}(N) \subseteq Z(\mathsf{Vec}(N))$ to α defined in Lemma 6.3.

Proof. Recall that as a $\mathsf{Vec}(N)$ bimodule, $\mathsf{Vec}(E) \cong \bigoplus_{g\in G}{}_g\mathsf{Vec}(N)$, where here ${}_g\mathsf{Vec}(N)$ can be viewed as the linear category of vector spaces graded by elements of the coset indexed by $g \in G$. Let us consider the section $G \ni g \mapsto \lambda_g \in E$ chosen above. We can identify the simple objects of ${}_g\mathsf{Vec}(N)$ as elements $\lambda_g n$ for $n \in N$. Furthermore, ${}_g\mathsf{Vec}(N) \cong \mathsf{Vec}(N)$ as a right N module, where $\lambda_g n' \triangleleft n := \lambda_g n'n$, but the left action of N on ${}_g\mathsf{Vec}(N)$ is given by $n \triangleright \lambda_g n' = \lambda_g(\lambda_g^{-1}n\lambda_g)n'$. In other words, the left action is twisted by the auto-equivalence $\lambda_g^{-1} \cdot \lambda_g \in \mathrm{Aut}(N)$. From the definition of the categorical action $\tilde{\alpha}$ [19, Equation 24] and the canonical copy of $\mathsf{Rep}(N) \subseteq Z(\mathsf{Vec}(N))$, the result follows. ■

Corollary 6.5. The canonical $\mathsf{Rep}(N)$-fibered enrichment of $\mathsf{Vec}(N)$ extends to $\mathsf{Vec}(E)$ if and only if $E \cong N \times G$. In this case, these extensions form a torsor over $H^1(G, Z(N))$.

Proof. Since the canonical fibered enrichment is fully faithful, by the previous lemma, we can lift the enrichment if and only if the categorical action $\alpha : \underline{G} \to \underline{\mathrm{Aut}}_\otimes^{br}(\mathsf{Rep}(G))$ is isomorphic to the trivial categorical action. This would imply, in particular, that each α_g is trivial, namely that $\pi(\lambda_g^{-1}n\lambda_g) = \pi(n)$ for all $n \in N$, $g \in G$, and $(\pi, V) \in \mathsf{Rep}(N)$. Applying this to the regular representation implies $\lambda_g^{-1}n\lambda_g = n$, and thus we have a decomposition $E \cong N \times_\omega G$ for some 2-cocycle $\omega \in Z^2(G, Z(N))$, where the action on the latter coefficient module is trivial. Furthermore, we see this 2-cocycle $\omega_{g,h}$ is precisely the $n_{g,h}$ associated to our choice of λ. But since the tensorator for the action α is given by $\mu_{g,h}^{(\pi,V)} = \pi(n_{g,h})$ by definition, we see that the action α is precisely the trivial action twisted by ω. Therefore, α is isomorphic to the trivial action precisely when $[\omega]$ is trivial in $H^2(G, Z(N))$, which happens precisely when E splits as $N \times G$. The final claim follows easily. ■

7 Application: Classification of G-Crossed Braidings

An interesting point of view we wish to advocate is that various sorts of structures on a G-graded extension can be equivalent to extensions of an enrichment on the base category. In particular, a braided fusion category can be canonically enriched over itself. In this section, our goal is to show that (equivalence classes of) G-crossed braidings on a G-graded fusion category $\mathcal{D}$, which restrict on the trivial graded component $\mathcal{C}$ to some fixed braiding, are exactly classified by extensions of the corresponding self-enrichment of $\mathcal{C}$ to $\mathcal{D}$.

While this proof essentially boils down to results in [12, 13, 19] using [22], we believe our point of view sheds new light on G-crossed braidings while simultaneously providing intuition for enriched extensions as being "something like a G-crossed braiding". We then apply our earlier results to give a classification of G-crossed braidings generalizing the results of Nikshych [40]. This allows us to classify G-crossed braidings on a G-graded fusion category $\mathcal{D}$ in terms of full subcategories of its Drinfeld center, satisfying some conditions.

7.1 The canonical self-enrichment and G-crossed braidings

Fix a braided fusion category $\mathcal{C}$ with braiding β.

Definition 7.1. The canonical *self-enrichment* $\mathcal{C} \to Z(\mathcal{C})$ is given by $c \mapsto (c, \beta_{c,-})$.

In Section 4.1, we defined a monoidal product on $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C})$ via lifting the product on $\underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ determined by the universal property discussed in Definition 2.14 via the forgetful 2-functor $\mathrm{Forget}_{\mathcal{C}}$, which automatically makes $\mathrm{Forget}_{\mathcal{C}}$ a monoidal 2-functor.

Recall from [19, Section 4.4], [12, Section 2.8], or [13, Sections 3.3 and 5.2] that the monoidal 2-groupoid of invertible $\mathcal{C}$-modules $\underline{\underline{\mathsf{Pic}}}(\mathcal{C})$ is also endowed with a monoidal product by lifting the relative product from $\underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$. In more detail, there is a canonical inclusion 2-functor $\underline{\underline{\mathsf{Pic}}}(\mathcal{C}) \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ that identifies the right action with the left action and one lifts the monoidal product to make this canonical inclusion into a monoidal 2-functor.

Observe now that this monoidal 2-functor $\underline{\underline{\mathsf{Pic}}}(\mathcal{C}) \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ naturally factors through $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C})$! Indeed, when one defines the right action on an invertible $\mathcal{C}$-module $\mathcal{M}$ as equal to the left action, we get an obvious $\mathcal{C}$-centered structure $\eta^{\mathcal{M}}$ given by the identity. Since the monoidal products on $\underline{\underline{\mathsf{Pic}}}(\mathcal{C})$ and $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C})$ were both lifted from $\underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$, we see we have a commuting triangle:

$$\begin{tikzcd}
\underline{\underline{\mathsf{Pic}}}(\mathcal{C}) \arrow[r, "(-\lhd\mathcal{C}:=\mathcal{C}\rhd-{,}\eta)"] \arrow[dr, "-\lhd\mathcal{C}:=\mathcal{C}\rhd-"'] & \underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C}) \arrow[d, "\mathrm{Forget}_{\mathcal{C}}"] \\
& \underline{\underline{\mathsf{BrPic}}}(\mathcal{C}).
\end{tikzcd} \tag{7.1}$$

The horizontal arrow in (7.1) is easily seen to be an equivalence of the underlying 2-groupoids, with inverse (up to equivalence) given by forgetting the right $\mathcal{C}$-action (cf. [12, Definition 2.12 and Remark 2.13]).

We now fix a braided fusion category $\mathcal{C}$ together with a G-extension $\mathcal{C} \subseteq \mathcal{D}$ as ordinary fusion categories corresponding to a monoidal 2-functor $\underline{\underline{\rho}} : \underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ from [19]. We are now ready to prove Theorem 1.3, which is (1) $\cong$ (4) of the following theorem.

Theorem 7.2. Fix a braided fusion category $\mathcal{C}$ and a G-extension $\mathcal{C} \subset \mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ as an ordinary fusion category. Let $\underline{\underline{\pi}} : \underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ be any monoidal 2-functor corresponding to $\mathcal{D}$ under Theorem 2.15. The following sets are in canonical bijection.

1. Lifts of the self $\mathcal{C}$-enrichment $\mathcal{F}^{z} : \mathcal{C} \to Z(\mathcal{C})$ to $\mathcal{D}$, that is, braided tensor functors $\widetilde{\mathcal{F}}^{z} : \mathcal{C} \to \mathsf{Rep}(G)' \subset Z(\mathcal{D})$ such that $\mathrm{Forget}_{\mathcal{C}} \circ \widetilde{\mathcal{F}}^{z} = i \circ \mathcal{F}^{z}$ where $i : Z(\mathcal{C}) \hookrightarrow Z_{\mathcal{C}}(\mathcal{D})$ is the canonical inclusion.
2. Lifts of $\underline{\underline{\pi}}$ to $\underline{\underline{\pi}}^{\mathcal{C}} : \underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C})$ such that $\mathrm{Forget}_{\mathcal{C}} \circ \underline{\underline{\pi}}^{\mathcal{C}} = \underline{\underline{\pi}}$ on the nose.
3. Lifts of $\mathcal{D}$ to $Z_{\mathcal{C}}(\mathcal{D})$ that agree with the reversed self-enrichment $\mathcal{F}^{Z}_{\mathrm{rev}} : \mathcal{C}^{\mathrm{rev}} \to Z(\mathcal{C})$, that is, tensor functors $F : \mathcal{D} \to Z_{\mathcal{C}}(\mathcal{D})$ such that $\mathrm{Forget}_{Z} \circ F = \mathrm{id}_{\mathcal{D}}$ on the nose and $F|_{\mathcal{C}} = i \circ \mathcal{F}^{Z}_{\mathrm{rev}}$.
4. The equivalence classes of G-crossed braidings on $\mathcal{D}$ (cf. Example 3.17).

Proof. $\underline{(1) \cong (2):}$ This is a special case of Theorem 4.5 with $\mathcal{V} = \mathcal{C}$ and $\mathcal{F}^{z} : \mathcal{C} \to Z(\mathcal{C})$ the self-enrichment.

$\underline{(1) \cong (3):}$ Observe that lifts $F : \mathcal{D} \to Z_{\mathcal{C}}(\mathcal{D})$ such that $\mathrm{Forget}_{Z} \circ F = \mathrm{id}_{\mathcal{D}}$ are in bijection with lifts $\widetilde{\mathcal{F}}^{z} : \mathcal{C} \to \mathsf{Rep}(G)' \subset Z(\mathcal{D})$ such that $\mathrm{Forget}_{\mathcal{C}} \circ \widetilde{\mathcal{F}}^{z} = i \circ \mathcal{F}^{z}$ by taking the inverse half-braiding.

(2) $\cong$ (4): In Example 3.18, we saw that the set (2) of lifts of $\underline{\underline{\pi}} : \underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$ to $\underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C})$ is the strict fiber $\mathsf{stFib}_{\underline{\underline{\pi}}}((\text{Forget}_{\mathcal{C}})_*)$ where $(\text{Forget}_{\mathcal{C}})_* : \text{Hom}(\underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C})) \to \text{Hom}(\underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C}))$ is post-composition with $\text{Forget}_{\mathcal{C}} : \underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C}) \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})$.

By essentially the same argument as in Example 3.17, the equivalence classes (4) of G-crossed braidings is equivalent to the 0-truncation of the homotopy fiber over $\mathcal{D}$ $\tau_0(\mathsf{hoFib}_{\mathcal{D}}(\text{Forget}_{\beta}))$ of the forgetful 2-functor $\text{Forget}_{\beta} : \mathsf{Ext}_{\mathsf{CrsBrd}}(G, \mathcal{C}) \to \mathsf{Ext}(G, \mathcal{C})$.

By Theorem 2.15, we have an equivalence of 2-groupoids $\text{Hom}(\underline{\underline{G}} \to \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})) \cong \mathsf{Ext}(G, \mathcal{C})$, and by [19, Theorem 7.12] (see also [13, Proposition 8.11 and Theorem 8.13]), there is an equivalence of 2-groupoids $\text{Hom}(\underline{\underline{G}}, \underline{\underline{\mathsf{Pic}}}(\mathcal{C})) \cong \mathsf{Ext}_{\mathsf{CrsBrd}}(G, \mathcal{C})$, where the latter denotes the 2-groupoid of G-crossed braided extensions of $\mathcal{C}$. Since the horizontal arrow in (7.1) above is a monoidal 2-equivalence, we see $\text{Hom}(\underline{\underline{G}}, \underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C})) \cong \mathsf{Ext}_{\mathsf{CrsBrd}}(G, \mathcal{C})$.

Putting it all together, we have a (weakly) commuting square of 2-functors

$$\begin{tikzcd}
\text{Hom}(\underline{\underline{G}}, \underline{\underline{\mathsf{BrPic}}}^{\mathcal{C}}(\mathcal{C})) \arrow[r, "\cong"] \arrow[d, "(\text{Forget}_{\mathcal{C}})_*"] & \mathsf{Ext}_{\mathsf{CrsBrd}}(G, \mathcal{C}) \arrow[d, "\text{Forget}_{\beta}"] \\
\text{Hom}(\underline{\underline{G}}, \underline{\underline{\mathsf{BrPic}}}(\mathcal{C})) \arrow[r, "\cong"] & \mathsf{Ext}(G, \mathcal{C}).
\end{tikzcd}$$

Since $\underline{\underline{\pi}}$ maps to $\mathcal{D}$ under the lower horizontal arrow, the homotopy fibers at $\underline{\underline{\pi}}$ and $\mathcal{D}$ are equivalent. We thus have canonical bijections

$$\begin{aligned}
\mathsf{stFib}_{\underline{\underline{\pi}}}((\text{Forget}_{\mathcal{C}})_*) &\cong \tau_0(\mathsf{hoFib}_{\underline{\underline{\pi}}}((\text{Forget}_{\mathcal{C}})_*)) \cong \tau_0(\mathsf{hoFib}_{\mathcal{D}}(\text{Forget}_{\beta})) \\
&\cong \left\{ \begin{array}{l} \text{Equivalence classes of } G\text{-} \\ \text{crossed braidings on } \mathcal{D} \end{array} \right\},
\end{aligned}$$

where τ_0 denotes the 0-truncations of the homotopy fibers. This completes the proof. ■

7.2 Classification of G-crossed braidings on a fixed G-graded fusion category

We can use Theorem 1.3 to obtain a classification of G-crossed braidings on a G-graded fusion category generalizing a similar style of classification by Nikshych of braidings on a fusion category [40]. Recall that if $A \in \mathcal{C}$ is an algebra object, a subcategory $\mathcal{D} \subseteq \mathcal{C}$ is called *transverse* to A if for all objects $d \in \mathcal{D}$, $\mathcal{C}(d \to A) = \mathcal{C}(d \to 1)$. Recall that a subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is called *replete* if for all triples (c, d, f) with $c \in \mathcal{C}$, $d \in \mathcal{D}$, and $f : c \to d$ an isomorphism, we have $c \in \mathcal{D}$ and $f \in \mathcal{D}(c \to d)$.

Theorem 7.3. Let $\mathcal{D} = \bigoplus \mathcal{D}_g$ be a faithfully G-graded fusion category and $\mathsf{Rep}(G) \subseteq Z(\mathcal{D})$ the canonical subcategory of the center. Then, G-crossed braidings on $\mathcal{D}$ are classified by full and replete fusion subcategories $\mathcal{A} \subseteq Z(\mathcal{D})$ satisfying the following properties:

1. $\mathcal{A} \subseteq \mathsf{Rep}(G)'$;
2. $|G|\mathrm{FPdim}(\mathcal{A}) = \mathrm{FPdim}(\mathcal{D})$;
3. $\mathcal{A}$ is transverse to $I(1)$, that is, for any $a \in \mathcal{A}$, $Z(\mathcal{D})(a \to I(1)) = Z(\mathcal{D})(a \to 1)$, where I is the right adjoint of the forgetful functor $\mathrm{Forget}_Z : Z(\mathcal{D}) \to \mathcal{D}$.

Proof. We have just shown that G-crossed braidings are classified by braidings on the trivial component $\mathcal{D}_e$, and a lift of this braided category to $Z(\mathcal{D})$. Given a braiding σ on $\mathcal{D}_e$ that lifts to the center of $\mathcal{D}$, this defines a full subcategory $\mathcal{A} \subseteq \mathsf{Rep}(G)' \cap Z(\mathcal{D})$, which is equivalent as a braided fusion category to $\mathcal{D}_e$ with braiding σ. By construction, $\mathrm{FPdim}(\mathcal{A}) = \mathrm{FPdim}(\mathcal{D}_e) = \mathrm{FPdim}(\mathcal{D})/|G|$ as desired. Furthermore, since the forgetful functor $\mathrm{Forget}_Z|_{\mathcal{A}}$ is fully faithful, $\mathcal{A}$ is transverse to $I(1)$.

Conversely, given a subcategory $\mathcal{A} \subseteq \mathsf{Rep}(G)' \cap Z(\mathcal{D})$ (condition (3)) that is transverse to $I(1)$ (condition (2)), $\mathrm{Forget}_Z|_{\mathcal{A}}$ is fully faithful. Since $\mathcal{A}$ centralizes $\mathsf{Rep}(G)$, $\mathrm{Forget}(\mathcal{A}) \subseteq \mathcal{D}_e$. If, moreover, condition (1) holds,

$$\mathrm{FPdim}(\mathcal{A}) = \mathrm{FPdim}(\mathrm{Forget}_Z(\mathcal{A})) = \frac{\mathrm{FPdim}(\mathcal{D})}{|G|} = \mathrm{FPdim}(\mathcal{D}_e),$$

$\mathrm{Forget}_Z|_{\mathcal{A}}$ is an equivalence. Thus, we can transport the half-braidings induced from $\mathcal{A}$ onto $\mathcal{D}_e$, to obtain a braiding that lifts to the center.

It is clear these two constructions are mutually inverse. ■

We note that a G-crossed braiding, by definition, is additional structure on a fusion category consisting of an entire categorical action by G and a family of natural isomorphisms satisfying complicated coherences. In the following two subsections, we apply Theorem 7.3 to provide a complete classification of H-crossed braidings on group theoretical categories of the form $\mathsf{Vec}(G,\omega)$ and $\mathsf{Rep}(G)$.

7.3 Example: $\mathsf{Vec}(G,\omega)$

First, we consider pointed categories $\mathcal{D} = \mathsf{Vec}(G,\omega)$ where G is a finite group and $\omega \in Z^3(G,\mathbb{C}^\times)$.

We recall the results of [38], which classifies fusion subcategories of $Z(\mathsf{Vec}(G,\omega))$. To state these results, given a normalized 3-cocycle ω, for any triple of elements $a, g, h \in G$, we define the function

$$\beta_a(g,h) := \frac{\omega(a,g,h)\omega(g,h,h^{-1}g^{-1}agh)}{\omega(g,g^{-1}ag,h)}.$$

Letting $C_G(a) = \{g \in G\ ga = ag\}$, then $\beta_a|_{C_G(a)\times C_G(a)} \in Z^2(C_G(a),\mathbb{C}^\times)$. Isomorphism classes of simple objects in $Z(\mathcal{C})$ are then classified by pairs (a,χ), where $a \in G$ is a representative of a conjugacy class and χ is an irreducible β_a-projective representation of $C_G(a)$ [8, 15].

Definition 7.4. Let $L, M \triangleleft G$ be commuting normal subgroups. A function $B : L\times M \to \mathbb{C}^\times$ is called an *ω-bicharacter* if

(1) $B(\ell, mn) = \beta_\ell^{-1}(m,n)B(\ell,m)B(\ell,n)$ for all $\ell \in L$ and $m,n \in M$;
(2) $B(k\ell, m) = \beta_m(k,\ell)B(k,m)B(\ell,m)$ for all $k,\ell \in L$ and $m \in M$.

An ω-bicharacter $B : L \times M \to \mathbb{C}^\times$ is called *G-invariant* if, moreover,

(3) $B(g^{-1}\ell g, m) = \beta_\ell(g,m)\beta_\ell(gm,g^{-1})\beta_\ell^{-1}(g,g^{-1})B(\ell,gmg^{-1})$ for all $g \in G$, $\ell \in L$, and $m \in M$.

We recall the following classification theorem.

Theorem 7.5 ([38, Theorem 5.11]). Full and replete fusion subcategories of $Z(\mathsf{Vec}(G,\omega))$ are classified by the following data:

- a pair L, M of commuting normal subgroups of G and
- a G-invariant ω-bicharacter $B : L \times M \to \mathbb{C}^\times$.

Given such an abstract fusion subcategory $\mathcal{A}$, the subgroup L is determined by the normal subgroup of G generated by the image of the forgetful functor, while M is determined by $\mathsf{Rep}(G/M) = \mathcal{A} \cap \mathsf{Rep}(G)$, where $\mathsf{Rep}(G)$ denotes the canonical copy of $\mathsf{Rep}(G) \subset Z(\mathsf{Vec}(G,\omega))$. See [38] for an explanation of the role of the bicharacter B.

We denote the subcategory associated to the above data as $\mathcal{S}(L,M,B)$. In this notation, the canonical subcategory $\mathsf{Rep}(G)$ is $\mathcal{S}(1,1,1)$ and the trivial subcategory Vec is $\mathcal{S}(1,G,1)$. We further recall the following facts from [38].

- $\mathrm{FPdim}(\mathcal{S}(L,M,B)) = |L|[G:M]$ [38, Lemma 5.9].
- $\mathcal{S}(L,M,B)' = \mathcal{S}(M,L,(B^{\mathrm{op}})^{-1})$ [38, Lemma 5.10].

- $\mathcal{S}(L,M,B) \subseteq \mathcal{S}(L',M',B')$ if and only if $L \subseteq L'$, $M' \subseteq M$ and $B|_{L\times M'} = B'|_{L\times M'}$ [38, Proposition 6.1].

Proposition 7.6. Suppose we have a faithful H-grading on $\mathsf{Vec}(G,\omega)$ given by a surjective homomorphism $\pi : G \to H$. Then, if $\mathsf{Vec}(G,\omega)$ admits an H-crossed braiding, $\ker(\pi) \subseteq Z(G)$. In this case, H-crossed braidings are classified by G-invariant ω-bicharacters $B : \ker(\pi) \times G \to \mathbb{C}^\times$.

Proof. It suffices to show that subcategories of $Z(\mathsf{Vec}(G,\omega))$ satisfying the conditions of 7.3 are precisely those of the form $\mathcal{S}(\ker(\pi), G, B)$, where B is an arbitrary G-invariant ω-bicharacter. Note that $\mathcal{S}(L,M,B)$ is transverse to $I(1)$ if and only if $\mathsf{Rep}(G/M) = \mathcal{S}(L,M,B) \cap \mathsf{Rep}(G) = \mathsf{Vec}$, since $I(1) = \mathcal{O}(G) \in \mathsf{Rep}(G)$ contains all the irreducible objects of $\mathsf{Rep}(G)$. Thus, $M = G$. Note this implies $L \le Z(G)$, since L must centralize M. Now, observe that $\mathcal{S}(L,G,B)$ centralizes $\mathsf{Rep}(H) = \mathsf{Rep}(G/\ker(\pi)) = \mathcal{S}(1,\ker(\pi),1)$ if and only if $\mathcal{S}(L,G,B) \le \mathcal{S}(\ker(\pi),1,1)$, which can be restated as $L \le \ker(\pi)$, $1 \le G$, and $B|_{L\times 1} = 1$, where the last follows automatically from the properties of ω-bicharacters. Finally, the third condition is $\mathrm{FPdim}(\mathcal{S}(L,G,B)) = |G|/|H| = |\ker(\pi)|$. However, $\mathrm{FPdim}(\mathcal{S}(L,G,B)) = |L|[G : G] = |L|$, and thus we must have $|L| = |\ker(\pi)|$. But since $L \le \ker(\pi)$, we must have equality, which concludes the proof. ∎

As a special case, we recover the following well known corollary.

Corollary 7.7. There is a unique G-crossed braiding on $\mathsf{Vec}(G,\omega)$.

Remark 7.8. Recall that a braiding on a G-graded fusion category $\mathcal{D}$ can be viewed as a G-crossed braiding together with an extra piece of data, namely a trivialization of the categorical action $\underline{G} \to \underline{\mathrm{Aut}_\otimes}(\mathcal{D})$. For example, when G is abelian, we have a unique G-crossed braiding on $\mathsf{Vec}(G)$, where the G action is by conjugation, and the G-braiding is the identity. However, we have several different braidings on $\mathsf{Vec}(G)$ that correspond to distinct trivializations of the conjugation action, which it is easy to show correspond to bicharacters on G.

7.4 Example: $\mathsf{Rep}(G)$

Now, we consider the case where $\mathcal{D} = \mathsf{Rep}(G)$, and we consider its center in terms of $Z(\mathsf{Vec}(G))$, where we can use the convenient description as above. In this case, the universal grading group is the dual group $\widehat{Z(G)}$. The copy of $\mathsf{Rep}(\widehat{Z(G)}) \cong \mathsf{Vec}(Z(G))$

sitting inside $Z(\mathsf{Vec}(G))$ is identified with the objects that are direct sums of objects $(z, 1)$ where $z \in Z(G)$ represents a conjugacy class, and 1 is the trivial representation of the centralizer subgroup of z (which is G).

Note that all (normal) subgroups of $\widehat{Z(G)}$ are of the form

$$H^{\perp} = \left\{\gamma \in \widehat{Z(G)} \middle| \gamma(h) = 1 \ \forall h \in H\right\}$$

for some $H \leq Z(G)$. Thus, faithful grading groups are given by quotients $\widehat{Z(G)}/H^{\perp}$, and $\mathsf{Rep}(\widehat{Z(G)}/H^{\perp}) \cong \mathsf{Vec}(H) \subseteq \mathsf{Vec}(Z(G))$.

We have the following result.

Proposition 7.9. For $H \leq Z(G)$, faithful $\widehat{Z(G)}/H^{\perp}$-crossed braidings on $\mathsf{Rep}(G)$ are classified by triples (L, M, B), such that

- $M \triangleleft G$ is normal such that $H \leq M$ and M/H is abelian;
- $L \triangleleft G$ is abelian and commutes with M; and
- $B : L \times M/H \to \mathbb{C}^{\times}$ is a nondegenerate G-invariant bicharacter.

Proof. Consider the faithful $\widehat{Z(G)}/H^{\perp}$-crossed braiding on $\mathsf{Rep}(G)$ corresponding to the fusion subcategory $\mathcal{S}(L, M, B) \subseteq Z(\mathsf{Vec}(G)) = Z(\mathsf{Rep}(G))$ under Theorem 7.3.

Step 1: The subgroups $L, M \leq G$ and the bicharacter B satisfy

- $M \triangleleft G$ is normal such that $H \leq M$ and $[M : H] = |L|$;
- $L \triangleleft G$ commutes with M; and
- $B : L \times M \to \mathbb{C}^{\times}$ is a G-invariant bicharacter such that $B|_{L\times H} = 1$ and the homomorphism $\widehat{B} : L \to \widehat{M},\ l \mapsto B(l, \cdot\,)$ is injective.

Proof of Step 1. The canonical copy of $\mathsf{Rep}(\widehat{Z(G)}/H^{\perp}) \cong \mathsf{Vec}(H)$ is given by the subcategory $\mathcal{S}(H, G, 1)$, whose centralizer is $\mathcal{S}(G, H, 1)$. Thus, $\mathcal{S}(L, M, B) \subseteq \mathcal{S}(H, G, 1)'$ if and only if $L \leq G, H \leq M$ and $B_{L\times H} = 1$. Now, the FP dimension condition is satisfied if and only if $\mathrm{FPdim}(\mathcal{S}(L, M, B)) = |L|[G : M] = [G : H]$, which happens if and only if $|L| = \frac{|M|}{|H|} = [M : H]$. Finally, $\mathcal{S}(L, M, B)$ is transverse to the Lagrangian algebra $I(1)$ (for $\mathsf{Rep}(G)$) if and only if the homomorphism $\widehat{B} : L \to \widehat{M},\ l \mapsto B(l, \cdot\,)$ is injective by [40, Lemma 5.1]. ■

Step 2: L and M/H are abelian, and $\widehat{B} : L \to \widehat{M/H}$ given by $\widehat{B}(l) := B(l, \cdot\,)$ is an isomorphism, which gives nondegeneracy of the bicharacter.

Proof of Step 2. By the first condition in Step 1, $|L| = |M/H|$. By the third condition, $\widehat{B} : L \to \widehat{M/H}$ is an injection. Thus, we have

$$|M/H| = |L| \leq |\widehat{M/H}| = |(M/H)/[M/H, M/H]| \leq |M/H|.$$

This forces the equality $|(M/H)/[M/H, M/H]| = |M/H|$, and thus M/H is abelian. Furthermore, this implies $|M/H| = |\widehat{M/H}|$, and thus the injective map $\widehat{B}$ is an isomorphism as claimed. ■

We now consider some examples.

Example 7.10. When $\widehat{Z(G)}/H^{\perp} = 1$ so that $H = 1 \leq Z(G)$, then we should recover braidings on $\mathsf{Rep}(G)$, which have been classified by [10] and again by [39], and indeed this is the case.

Example 7.11. Consider the case $H = Z(G)$, so that the grading on $\mathsf{Rep}(G)$ is the universal grading. Then, choosing $M = Z(G)$ and $L = 1$ and $B = 1$, we obtain the usual braiding on $\mathsf{Rep}(G)$, viewed as a G-crossed braiding.

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